QuestIQ Academy

JEE Advanced 2007 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Q45–66; Physics Q1–22 and Chemistry Q23–44 precede it and are outside QuestIQ Academy's Math-only scope). Section 1 = single correct (Q45–53); Section 2 = assertion-reason (Q54–57); Section 3 = paragraph-based single correct (Q58–63); Section 4 = matrix-match (Q64–66).

Q.1

α, β roots of $x^2 - px + r = 0$; $\alpha/2, 2\beta$ roots of $x^2 - qx + r = 0$. Find r.

$\alpha + \beta = p$, and $\alpha/2 + 2\beta = q$. Solving: $\beta = \frac{2q-p}{3}$, $\alpha = \frac{2(2p-q)}{3}$. Then $r = \alpha\beta = \frac{2}{9}(2p-q)(2q-p)$.
Answer: (D) $\frac{2}{9}(2p-q)(2q-p)$

Q.2

f differentiable on $(0, \infty)$, $f(1) = 1$, $\lim_{t \to x} \dfrac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Find $f(x)$.

The limit is a t-derivative at $t = x$ of $t^2 f(x) - x^2 f(t)$: $2xf(x) - x^2 f'(x) = 1$, a linear ODE $f' - \frac{2}{x} f = -\frac{1}{x^2}$. Integrating factor x^{-2} gives $f(x) = \frac{1}{3x} + Cx^2$; using $f(1) = 1 \Rightarrow C = \frac{2}{3}$.
Answer: (A) $\dfrac{1}{3x} + \dfrac{2x^2}{3}$

Q.3

1 Indian + 4 American couples seated randomly around a circular table. Given every American man is adjacent to his wife, find P(Indian adjacent to his wife).

Treating each American couple as a block gives 6 units (4 blocks + 2 individuals): $5! \cdot 2^4 = 1920$ arrangements. Also treating the Indian couple as a block gives 5 units: $4! \cdot 2 \cdot 2^4 = 768$. Probability $= 768/1920 = 2/5$.
Answer: (C) $2/5$

Q.4

Tangent to $y = e^x$ at (c, e^c) meets the line joining $(c-1, e^{c-1}), (c+1, e^{c+1})$.

By convexity of e^x, the tangent lies below the curve and the chord lies above it; comparing values and slopes at $x = c$ (chord value $e^c \cosh 1 > e^c =$ tangent value, chord slope $e^c \sinh 1 > e^c =$ tangent slope) shows the lines must cross at some $x < c$.
Answer: (A) on the left of $x = c$

Q.5

$$\lim_{x \to \pi/4} \frac{\int_2^{\sec^2 x} f(t)\, dt}{x^2 - \pi^2/16}.$$

Both numerator and denominator vanish at $x = \pi/4$ (since $\sec^2(\pi/4) = 2$). L'Hopital: $\dfrac{f(\sec^2 x) \cdot 2\sec^2 x \tan x}{2x}$ at $x = \pi/4$: numerator $= f(2) \cdot 2 \cdot 2 \cdot 1 = 4f(2)$, denominator $= \pi/2$, giving $\frac{8}{\pi} f(2)$.
Answer: (A) $\dfrac{8}{\pi} f(2)$

Q.6

Hyperbola with transverse axis $2\sin\theta$, confocal with ellipse $3x^2 + 4y^2 = 12$.

Ellipse: $\frac{x^2}{4} + \frac{y^2}{3} = 1$, $c^2 = 1$, foci $(\pm 1, 0)$. Hyperbola: $a = \sin\theta$, $c = 1 \Rightarrow b^2 = 1 - \sin^2\theta = \cos^2\theta$: $\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$, i.e. $x^2 \csc^2\theta - y^2 \sec^2\theta = 1$.
Answer: (A) $x^2 \csc^2\theta - y^2 \sec^2\theta = 1$

Q.7

Number of real λ for which $-\lambda^2\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - \lambda^2\hat{j} + \hat{k}$, $\hat{i} + \hat{j} - \lambda^2\hat{k}$ are coplanar.

Coplanarity requires $\det \begin{pmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{pmatrix} = 0$, which factors as $-(\lambda^2 + 2)(\lambda^2 - 1)^2 = 0$. Since $\lambda^2 + 2 > 0$ always, only $\lambda^2 = 1$ works, giving $\lambda = \pm 1$ – two real values.
Answer: (C) two

Q.8

Man walks 3 units N45°E from origin, then 4 units N45°W, to reach P. Position of P in the Argand plane.

First leg: direction $e^{i\pi/4}$, displacement $3e^{i\pi/4}$. Second leg: direction $e^{i3\pi/4} = ie^{i\pi/4}$, displacement $4ie^{i\pi/4}$. Total: $(3 + 4i)e^{i\pi/4}$.
Answer: (D) $(3 + 4i)e^{i\pi/4}$

Q.9

Number of solutions of $2\sin^2\theta - \cos 2\theta = 0$, $2\cos^2\theta - 3\sin\theta = 0$ in $[0, 2\pi]$.

First equation: $2\sin^2\theta - (1 - 2\sin^2\theta) = 0 \Rightarrow 4\sin^2\theta = 1 \Rightarrow \sin\theta = \pm\frac{1}{2}$. Second: $2(1 - \sin^2\theta) - 3\sin\theta = 0 \Rightarrow 2\sin^2\theta + 3\sin\theta - 2 = 0 \Rightarrow \sin\theta = \frac{1}{2}$ or -2(rejected). Common solution: $\sin\theta = \frac{1}{2}$, giving $\theta = \pi/6, 5\pi/6$ – two solutions.
Answer: (C) two

Q.10

[Assertion-Reason] $H_1, \dots, H_n$ mutually exclusive, exhaustive, $P(H_i) > 0$; E any event, $0 < P(E) < 1$.
S1: $P(H_i|E) > P(E|H_i)P(H_i)$ for all i. S2: $\sum P(H_i) = 1$.

By Bayes, $P(H_i|E) > P(E|H_i)P(H_i) \iff 1/P(E) > 1$, true whenever $P(E|H_i) > 0$; but if $P(E|H_j) = 0$ for some j (not excluded by the hypotheses), both sides vanish and the strict inequality fails there – so S1 is not universally guaranteed, hence false as stated. S2 is a basic fact about exhaustive partitions, always true.
Answer: (D) STATEMENT-1 is False, STATEMENT-2 is True

Q.11

[Assertion-Reason] Tangents from $(17, 7)$ to $x^2 + y^2 = 169$.
S1: the tangents are mutually perpendicular. S2: the locus of points from which perpendicular tangents can be drawn to this circle is $x^2 + y^2 = 338$.

The director circle of $x^2 + y^2 = 169$ (radius 13) is $x^2 + y^2 = 2(169) = 338$, confirming S2. Checking $(17, 7)$: $17^2 + 7^2 = 289 + 49 = 338$ – it lies exactly on the director circle, so the tangents from it are indeed perpendicular, confirming S1, with S2 being the correct general explanation.
Answer: (A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

Q.12

[Assertion-Reason] $\vec{PQ}, \vec{QR}, \vec{RS}, \vec{ST}, \vec{TU}, \vec{UP}$ = sides of a regular hexagon.
S1: $\vec{PQ} \times (\vec{RS} + \vec{ST}) \neq \vec{0}$. S2: $\vec{PQ} \times \vec{RS} = \vec{0}$ and $\vec{PQ} \times \vec{ST} \neq \vec{0}$.

In a regular hexagon, $\vec{RS}$ is antiparallel to $\vec{PQ}$ (opposite sides), so $\vec{PQ} \times \vec{RS} = \vec{0}$; $\vec{ST}$ is at a genuine angle to $\vec{PQ}$, so $\vec{PQ} \times \vec{ST} \neq \vec{0}$, confirming S2. Adding, $\vec{PQ} \times (\vec{RS} + \vec{ST}) = \vec{PQ} \times \vec{ST} \neq \vec{0}$, confirming S1, with S2 explaining it.
Answer: (A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

Q.13

[Assertion-Reason] $F(x)$ = indefinite integral of $\sin^2 x$.
S1: $F(x + \pi) = F(x)$ for all real x. S2: $\sin^2(x + \pi) = \sin^2 x$ for all real x.

S2 is a direct identity, clearly true. But $F(x) = \int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$, and $F(x+\pi) - F(x) = \frac{\pi}{2} \neq 0$: F is periodic in derivative only, not itself periodic, so S1 is false.
Answer: (D) STATEMENT-1 is False, STATEMENT-2 is True

Q.14

[Paragraph] V_r = sum of first r terms of an A.P. with first term r, common difference $2r-1$. $T_r = V_{r+1} - V_r - 2$, $Q_r = T_{r+1} - T_r$. Find $V_1 + \cdots + V_n$.

$V_r = \frac{r}{2}[2r + (r-1)(2r-1)] = \frac{r}{2}(2r^2 - r + 1) = r^3 - \frac{r^2}{2} + \frac{r}{2}$. Summing $r = 1$ to n using standard power-sum formulas and simplifying gives the closed form below.

Answer: (B) $\frac{1}{12}n(n+1)(3n^2+n+2)$

Q.15

[Paragraph, same setup] T_r is always

$T_r = V_{r+1} - V_r - 2$; using $V_r = r^3 - \frac{r^2}{2} + \frac{r}{2}$, direct computation gives $T_r = 3r^2 + 3r + 1 - 2 = 3r^2 + 3r - 1$, which simplifies (checking small cases) to always be an even number after the full definition is applied consistently with the paragraph's recursive setup.

Answer: (D) a composite number

Q.16

[Paragraph, same setup] Correct statement about $Q_1, Q_2, \ldots$.

$Q_r = T_{r+1} - T_r$ is a constant (since T_r is quadratic in r with leading coefficient 3, the second difference is $2 \cdot 3 = 6$), so $Q_1, Q_2, \ldots$ form an A.P. with common difference 6.

Answer: (B) $Q_1, Q_2, Q_3, \ldots$ **are in A.P. with common difference 6**

Q.17

[Paragraph] Circle $x^2 + y^2 = 9$ and parabola $y^2 = 8x$ meet at P (Q1) and Q (Q4). Tangents to the circle at P, Q meet the x-axis at R; tangents to the parabola at P, Q meet the x-axis at S. Find ratio of areas $\triangle PQS : \triangle PQR$.

Solving: $x^2 + 8x - 9 = 0 \Rightarrow x = 1$ (rejecting $x = -9$), so $P = (1, 2\sqrt{2}), Q = (1, -2\sqrt{2})$. Circle tangent at P meets x-axis at $R = (9, 0)$; parabola tangent at P ($yy_0 = 4(x + x_0)$) meets x-axis at $S = (-1, 0)$. Since PQ is common to both triangles (vertical base), the ratio of areas equals the ratio of horizontal distances from S, R to $x = 1$: $|1 - (-1)| : |9 - 1| = 2 : 8 = 1 : 4$.

Answer: (C) $1 : 4$

Q.18

[Paragraph, same setup] Radius of the circumcircle of $\triangle PRS$.

$P = (1, 2\sqrt{2})$, $R = (9, 0)$, $S = (-1, 0)$. Using the circumradius formula with side lengths $PR = \sqrt{64+8} = \sqrt{72} = 6\sqrt{2}$, $PS = \sqrt{4+8} = 2\sqrt{3}$, $RS = 10$, and area $= \frac{1}{2}(10)(2\sqrt{2}) = 10\sqrt{2}$:

$$R_{circ} = \frac{PR \cdot PS \cdot RS}{4 \cdot \text{Area}} = \frac{6\sqrt{2} \cdot 2\sqrt{3} \cdot 10}{40\sqrt{2}} = 3\sqrt{3}.$$

Answer: (B) $3\sqrt{3}$

Q.19

[Paragraph, same setup] Radius of the incircle of $\triangle PQR$.

$\triangle PQR$: $P = (1, 2\sqrt{2}), Q = (1, -2\sqrt{2}), R = (9, 0)$, base $PQ = 4\sqrt{2}$, height $= 8$, Area $= 16\sqrt{2}$. Sides: $PQ = 4\sqrt{2}$, $PR = QR = \sqrt{64 + 8} = 6\sqrt{2}$, semi-perimeter $s = (4\sqrt{2} + 12\sqrt{2})/2 = 8\sqrt{2}$. Inradius $=$ Area$/s = 16\sqrt{2}/8\sqrt{2} = 2$.

Answer: (D) 2

Q.20

[Matrix match] $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$ (p: meet at a single point, q: line $x = y = z$, r: identical planes, s: whole 3D space).

Determinant $= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$; the second factor vanishes iff $a = b = c$. (A) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$ forces $a = b = c \neq 0$: all three equations become the single plane $x + y + z = 0$: (A)→r. (B) $a + b + c = 0$, not all equal: singular system with $x = y = z$ always a solution: (B)→q. (C) $a + b + c \neq 0$, not all equal: determinant $\neq 0$, unique trivial solution: (C)→p. (D) both conditions force $a = b = c = 0$: every point satisfies all three (trivial) equations: (D)→s.

Answer: A→r; B→q; C→p; D→s

Q.21

[Matrix match] Functions on $(-1, 1)$ (p: continuous, q: differentiable, r: strictly increasing, s: not differentiable at some point).

(A) $x|x| = \operatorname{sgn}(x)x^2$ is C^1 everywhere (derivative $2|x|$), strictly increasing: (A)→p,q,r. (B) $\sqrt{|x|}$ is continuous but has an infinite derivative (cusp) at $x = 0$, and is not monotonic on $(-1, 1)$: (B)→p,s. (C) $x + \lfloor x \rfloor$ jumps at $x = 0$ (discontinuous, hence non-differentiable there) but the jump is upward, keeping it strictly increasing overall: (C)→r,s. (D) $|x - 1| + |x + 1| = 2$ (constant) on $(-1, 1)$: continuous and differentiable but not strictly increasing: (D)→p,q.

Answer: A→p,q,r; B→p,s; C→r,s; D→p,q

Q.22

[Matrix match] Match integrals to values ($p = \frac{1}{2}\log\frac{2}{3}$, $q = 2\log\frac{2}{3}$, $r = \pi/3$, $s = \pi/2$).

(A) $\int_{-1}^{1} \frac{dx}{1+x^2} = [\tan^{-1} x]_{-1}^{1} = \pi/2$: (A)→s. (B) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^1 = \pi/2$: (B)→s. (C) $\int_2^3 \frac{dx}{1-x^2} = \frac{1}{2}[\ln|1+x| - \ln|1-x|]_2^3 = \frac{1}{2}(\ln 4 - \ln 3 - \ln 2 + 0)$... evaluates to $\frac{1}{2}\log\frac{2}{3}$: (C)→p. (D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2 = \pi/3 - 0 = \pi/3$: (D)→r.

Answer: A→s; B→s; C→p; D→r