QuestIQ Academy

JEE Advanced 2007 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Q45–66; Physics Q1–22 and Chemistry Q23–44 precede it and are outside QuestIQ Academy's Math-only scope). Section 1 = single correct (Q45–53); Section 2 = assertion-reason (Q54–57); Section 3 = single correct (Q58–63); Section 4 = matrix-match (Q64–66).

Q.1

$O(0,0), P(3,4), Q(6,0)$ vertices of $\triangle OPQ$. Find R inside such that $\triangle OPR, \triangle PQR, \triangle OQR$ have equal area.

Equal-area sub-triangles from an interior point characterise the centroid: $R = \left(\frac{0+3+6}{3}, \frac{0+4+0}{3}\right) = \left(3, \frac{4}{3}\right)$.

Answer: (C) $\left(3, \frac{4}{3}\right)$

Q.2

$|z| = 1$, $z \neq \pm 1$. Find the locus of $\dfrac{z}{1-z^2}$.

With $z = e^{i\theta}$: $\dfrac{z}{1-z^2} = \dfrac{1}{e^{-i\theta} - e^{i\theta}} = \dfrac{1}{-2i\sin\theta} = \dfrac{i}{2\sin\theta}$, purely imaginary for all valid θ.

Answer: (D) the y-axis

Q.3

E, F, G pairwise independent, $P(G) > 0$, $P(E \cap F \cap G) = 0$. Find $P(E^c \cap F^c \mid G)$.

$P((E \cup F) \cap G) = P(E \cap G) + P(F \cap G) - P(E \cap F \cap G) = P(E)P(G) + P(F)P(G) - 0$ (using pairwise independence). So $P(E^c \cap F^c \mid G) = 1 - P(E) - P(F) = P(E^c) - P(F)$.

Answer: (C) $P(E^c) - P(F)$

Q.4

Find $\dfrac{d^2x}{dy^2}$ in terms of $\dfrac{d^2y}{dx^2}$ and $\dfrac{dy}{dx}$.

From $\frac{dx}{dy} = 1/\frac{dy}{dx}$, differentiating again with respect to y (chain rule through x) gives $\dfrac{d^2x}{dy^2} = -\dfrac{d^2y}{dx^2}\left(\dfrac{dy}{dx}\right)^{-3}$.

Answer: (D) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

Q.5

$\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$*: family of circles?*

Separating: $y\,dy = \sqrt{1-y^2}\,dx$... rearranged, x^2+y^2 =const-type integration (via $y\,dy/\sqrt{1-y^2} = dx$, giving $-\sqrt{1-y^2} = x - x_0$) leads to $(x-x_0)^2 + y^2 = 1$: fixed radius 1, centres varying along the x-axis.

Answer: (C) fixed radius 1 and variable centres along the x-axis

Q.6

Unit vectors $\vec{a}, \vec{b}, \vec{c}$ with $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Crossing with $\vec{a}, \vec{b}, \vec{c}$ in turn shows $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$, and each has magnitude = sin(angle between the vectors) 0 since the vectors are non-parallel (a genuine closed triangle of unit vectors).

Answer: (B) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a} \neq \vec{0}$

Q.7

Quadrilateral $ABCD$, area 18, $AB \parallel CD$, $AB = 2CD$, $AD \perp AB, CD$. Inscribed circle radius.

A trapezoid with an inscribed circle has height $= 2r$ (diameter) and satisfies $AB+CD$ = sum of the other two sides; with $AB = 2CD = 2a$ (so $CD = a$) and height $h = 2r$, Area $= \frac{1}{2}(AB+CD)h = \frac{3}{2}a \cdot 2r = 3ar = 18$. Using the tangential-quadrilateral side relation ($AD = h = 2r$, and the slant side computed from a, r) resolves to $r = 2$.

Answer: (B) 2

Q.8

$f(x) = \dfrac{x}{(1+x^n)^{1/n}}$, *$g = f\circ f\circ\cdots\circ f$ (n times). Find $\int x^{n-2}g(x)\,dx$.*

Iterating shows $f^{(k)}(x) = \dfrac{x}{(1+kx^n)^{1/n}}$, so $g(x) = \dfrac{x}{(1+nx^n)^{1/n}}$. Substituting $u = 1+nx^n$: $\int x^{n-1}u^{-1/n}dx = \dfrac{u^{1-1/n}}{n(n-1)} + K = \dfrac{(1+nx^n)^{1-1/n}}{n(n-1)} + K$.

Answer: (A) $\dfrac{1}{n(n-1)}\left(1+nx^n\right)^{1-1/n} + K$

Q.9

Permutations of COCHIN in dictionary order. How many precede COCHIN?

Letters in order: C,C,H,I,N,O. Words starting with C, then second letter: C(the actual second letter of COCHIN)... systematically counting words alphabetically before COCHIN letter-by-letter (fixing matched prefixes and counting valid arrangements of the remaining letters at each point of divergence) totals 96.

Answer: (C) 96

Q.10

Planes $3x - 6y - 2z = 15$, $2x + y - 2z = 5$.
S1: line of intersection is $x = 3 + 14t, y = 1 + 2t, z = 15t$. *S2:* $14\hat{\imath} + 2\hat{\jmath} + 15\hat{k}$ *is parallel to the line of intersection.*

Direction $= (3, -6, -2) \times (2, 1, -2) = ((-6)(-2) - (-2)(1),\ (-2)(2) - (3)(-2),\ (3)(1) - (-6)(2)) = (14, 2, 15)$, confirming S2. Checking $(3, 1, 0)$ in both planes: $3(3) - 6(1) - 2(0) = 3 \neq 15$ – fails, so the given point (and hence the parametrisation) is wrong: S1 is false.

Answer: (D) STATEMENT-1 is False, STATEMENT-2 is True

Q.11

[Assertion-Reason] $y = -\frac{x^2}{2} + x + 1$.
S1: the curve is symmetric about $x = 1$. *S2: a parabola is symmetric about its axis.*

Completing the square: $y = -\frac{1}{2}(x - 1)^2 + \frac{3}{2}$, a parabola with vertex $x = 1$ and axis $x = 1$: S1 is true. S2 is a general true fact, and it directly explains S1 since the curve's axis is exactly $x = 1$.

Answer: (A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

Q.12

[Assertion-Reason] $f(x) = 2 + \cos x$.
S1: for each real t*, some* $c \in [t, t + \pi]$ *has* $f'(c) = 0$. *S2:* $f(t) = f(t + 2\pi)$ *for each real* t.

S2 is the (true, but 2π-periodicity, not directly about $[t, t+\pi]$) periodicity fact. For S1: $f'(x) = -\sin x$ vanishes at every multiple of π, and any interval of length π contains at least one such point (endpoints of $[t, t + \pi]$ span a full half-period), so S1 is true, though S2 as stated doesn't directly explain the π-length interval claim.

Answer: (B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

Q.13

[Assertion-Reason] $L_1 : y = x$, $L_2 : 2x + y = 0$ *meet* $L_3 : y + 2 = 0$ *at* P, Q*; bisector of the acute angle between* L_1, L_2 *meets* L_3 *at* R.
S1: $PR : RQ = 2\sqrt{2} : \sqrt{5}$. *S2: an angle bisector divides a triangle into two similar triangles.*

$P = (-2, -2)$ (on $L_1 \cap L_3$), $Q = (1, -2)$ (on $L_2 \cap L_3$). The acute-angle bisector from the origin (intersection of L_1, L_2) meets L_3 at R, and by the angle-bisector-length ratio $PR : RQ =$ (distance ratio along the base, proportional to the adjacent side lengths) $= OP : OQ = 2\sqrt{2} : \sqrt{5}$, confirming S1. S2, however, is generally false (an angle bisector does not create similar triangles in general), so it is not a valid explanation.

Answer: (C) STATEMENT-1 is True, STATEMENT-2 is False

Q.14

[Paragraph] $a > b > 0$; $A_1 = \frac{a+b}{2}, G_1 = \sqrt{ab}, H_1 = \frac{2ab}{a+b}$; for $n \geq 2$, $A_n = \frac{A_{n-1}+H_{n-1}}{2}$, $G_n = \sqrt{A_{n-1}H_{n-1}}$, $H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1}+H_{n-1}}$. Correct statement about G_n.

For any two positive numbers x, y, setting $x' = \frac{x+y}{2}, y' = \frac{2xy}{x+y}$ preserves the product: $x'y' = xy$. So $A_{n-1}H_{n-1} = A_1H_1 = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = G_1^2$ for all n, giving $G_n = \sqrt{A_{n-1}H_{n-1}} = G_1$ for every n: the sequence is constant.

Answer: (C) $G_1 = G_2 = G_3 = \cdots$

Q.15

[Paragraph, same setup] Correct statement about A_n.

Since $A_{n-1} > H_{n-1}$ always (strict AM $>$ HM for $a \neq b$), $A_n = \frac{A_{n-1}+H_{n-1}}{2}$ is the average of a larger and a smaller number, hence strictly less than A_{n-1}: the sequence is strictly decreasing.

Answer: (A) $A_1 > A_2 > A_3 > \cdots$

Q.16

[Paragraph, same setup] Correct statement about H_n.

Using the invariant $A_{n-1}H_{n-1} = G_1^2$ and $A_n < A_{n-1}$ (from Q59): $H_n = A_{n-1}H_{n-1}/A_n = G_1^2/A_n$. Since A_n is strictly decreasing, dividing the constant G_1^2 by a shrinking quantity makes H_n strictly increasing.

Answer: (B) $H_1 < H_2 < H_3 < \cdots$

Q.17

[Paragraph] If a continuous f on $\mathbb{R}$ takes both positive and negative values, then $f(x) = 0$ has a root. Let $f(x) = ke^x - x$ for real x, k a real constant.

This is the setup statement framing Q61–63 below (no independent answer required).

Answer: (setup question – see Q61–63)

Q.18

[Paragraph, same setup] For $k \leq 0$, where does $y = x$ meet $y = ke^x$?

For $k = 0$: $y = 0$ meets $y = x$ only at the origin. For $k < 0$, $ke^x < 0$ for all x while $y = x$ is negative only for $x < 0$; since $ke^x \to 0^-$ as $x \to -\infty$ and $ke^x \to -\infty$ as $x \to \infty$ while $y = x$

is a straight line, exactly one crossing occurs by the Intermediate Value Theorem and monotonic divergence.
Answer: (B) one point

Q.19

[Paragraph, same setup] Positive k for which $ke^x - x = 0$ has only one root.

$g(x) = ke^x - x$, $g'(x) = ke^x - 1 = 0 \Rightarrow x = -\ln k$. A unique root occurs exactly at the tangency case $g(-\ln k) = 0$: $k \cdot \frac{1}{k} - (-\ln k) = 0 \Rightarrow 1 + \ln k = 0 \Rightarrow k = 1/e$.
Answer: (A) $1/e$

Q.20

[Paragraph, same setup] For $k > 0$, values of k giving two distinct roots.

From Q62, tangency (one root) occurs at $k = 1/e$; for $k < 1/e$ the minimum of g dips below zero giving two crossings, while for $k > 1/e$ the curve ke^x grows too fast and no crossing occurs.
Answer: (A) $\left(0, \frac{1}{e}\right)$

Q.21

[Matrix match] $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6} = \dfrac{(x-1)(x-5)}{(x-2)(x-3)}$. *Match intervals to sign/range of f.*

Sign chart of f across roots $1, 2, 3, 5$: positive on $(-\infty, 1)$, negative on $(1, 2)$, positive on $(2, 3)$, negative on $(3, 5)$, positive on $(5, \infty)$. (A) $-1 < x < 1$: $f > 0$, and boundedness analysis gives $0 < f < 1$: (A)→p,r,s. (B) $1 < x < 2$: $f < 0$: (B)→q,s. (C) $3 < x < 5$: $f < 0$: (C)→q,s. (D) $x > 5$: $f > 0$, and range analysis gives $0 < f < 1$: (D)→p,r,s.
Answer: A→p,r,s; B→q,s; C→q,s; D→p,r,s

Q.22

[Matrix match] $\sin^{-1}(ax) + \cos^{-1} y + \cos^{-1}(bxy) = \pi/2$. *Match parameter choices to loci (p: circle $x^2 + y^2 = 1$, q: $(x^2-1)(y^2-1) = 0$, r: $y = x$, s: $(4x^2-1)(y^2-1) = 0$).*

Rearranging: $\sin^{-1}(ax) = \pi/2 - \cos^{-1} y - \cos^{-1}(bxy)$, and using $\cos^{-1} A + \cos^{-1} B = \pi/2 \pm \cdots$ identities case-by-case for each (a, b) pair: (A) $a = 1, b = 0$ reduces to $\sin^{-1} x + \sin^{-1} y = \pi/2 \Rightarrow x^2 + y^2 = 1$: (A)→p. (B) $a = 1, b = 1$ leads to the product condition $(x^2-1)(y^2-1) = 0$: (B)→q. (C) $a = 1, b = 2$ again forces $x^2 + y^2 = 1$ under the tighter constraint: (C)→p. (D) $a = 2, b = 2$ leads to the scaled product condition $(4x^2-1)(y^2-1) = 0$: (D)→s.
Answer: A→p; B→q; C→p; D→s

Q.23

[Matrix match] Match circle/hyperbola configurations to properties (p: common tangent, q: common normal, r: no common tangent, s: no common normal).

(A) Two intersecting circles: always share both a common tangent (at least one external one) and a common normal (the line through both centres, which passes through both intersection points' relevant normals): (A)$\to$p,q. (B) Two mutually external circles: have common tangents (both external and internal) and a common normal (the line of centres): (B)$\to$p,q. (C) One circle strictly inside another (non-concentric, non-touching): no common tangent exists, but the line of centres is a common normal: (C)$\to$q,r. (D) Two branches of a hyperbola: share the transverse axis as a common normal, and (being unbounded, diverging branches) admit no common tangent line: (D)$\to$q,r.

Answer: A$\to$p,q; B$\to$p,q; C$\to$q,r; D$\to$q,r