QuestIQ Academy

JEE Advanced 2010 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = paragraph-based single correct; Section 4 = single-digit integer.

Q.1

Angles A, B, C of a triangle are in AP; a, b, c opposite sides. Find $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$.

AP forces $B = 60^\circ$. By the sine rule, $a/c = \sin A/\sin C$, so the expression is $2\sin A\cos C + 2\sin C\cos A = 2\sin(A+C) = 2\sin(180^\circ - B) = 2\sin 60^\circ = \sqrt{3}$.
Answer: (D) $\sqrt{3}$

Q.2

Plane containing $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, perpendicular to the plane containing $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$.

The second plane has normal $(3,4,2)\times(4,2,3) = (8,-1,-10)$. Our plane's normal is perpendicular to both $(2,3,4)$ and this vector: $(2,3,4)\times(8,-1,-10) \propto (1,-2,1)$. Through the origin: $x - 2y + z = 0$.
Answer: (C) $x - 2y + z = 0$

Q.3

$\omega \neq 1$ a cube root of unity; die thrown 3 times giving r_1, r_2, r_3. Find $P(\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0)$.

The sum vanishes iff $\{r_1, r_2, r_3 \bmod 3\} = \{0,1,2\}$ (using $1 + \omega + \omega^2 = 0$). Each residue class has 2 die faces, giving $3! \times 2^3 = 48$ favourable outcomes out of 216: $48/216 = 2/9$.
Answer: (C) 2/9

Q.4

P, Q, R, S have position vectors $-2\hat{\imath} - \hat{\jmath}, 4\hat{\imath}, 3\hat{\imath} + 3\hat{\jmath}, -3\hat{\imath} + 2\hat{\jmath}$. Classify quadrilateral $PQRS$.

$\vec{PQ} = (6,1) = \vec{SR}$, confirming a parallelogram. $\vec{PQ}\cdot\vec{QR} = -3 \neq 0$ (not a rectangle), and $|\vec{PQ}| = \sqrt{37} \neq \sqrt{10} = |\vec{QR}|$ (not a rhombus).
Answer: (A) parallelogram, neither rhombus nor rectangle

Q.5

Number of 3×3 0/1-matrices A for which $A(x,y,z)^T = (1,0,0)^T$ has exactly two distinct (real) solutions.

Solutions of a linear system form an affine subspace: over $\mathbb{R}$ this has 0, exactly 1, or infinitely many points – never exactly two.

Answer: (A) 0

Q.6

$\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt.$

For small t, the integrand $\approx t^2/4$ (using $\ln(1+t) \approx t$ and $t^4+4 \approx 4$), so $\int_0^x t^2/4\,dt = x^3/12$, giving the limit 1/12.

Answer: (B) $1/12$

Q.7

$p \neq 0, p^3 \neq \pm q$; α, β nonzero complex with $\alpha+\beta = -p, \alpha^3+\beta^3 = q$. Quadratic with roots $\alpha/\beta, \beta/\alpha$.

$\alpha\beta = (p^3+q)/(3p)$ (from the sum-of-cubes identity). The roots' sum is $p^2/(\alpha\beta) - 2 = (p^3 - 2q)/(p^3+q)$ and product 1, giving $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$.

Answer: (B)

Q.8

$f = e^{x^2} + e^{-x^2}$, $g = xe^{x^2} + e^{-x^2}$, $h = x^2e^{x^2} + e^{-x^2}$ on $[0,1]$; a, b, c their absolute maxima.

Each of f', g', h' is ≥ 0 on $[0,1]$ (direct sign check), so all three functions are increasing, attaining their maxima at $x = 1$, where $f(1) = g(1) = h(1) = e + 1/e$.

Answer: (D) $a = b = c$

Q.9

A, B on $y^2 = 4x$; axis of parabola touches the circle with diameter AB, radius r. Possible slope(s) of AB.

With $A = (a^2, 2a), B = (b^2, 2b)$, tangency to the x-axis gives $|a+b| = r$; the slope is $2/(a+b)$, so slope $= \pm 2/r$.

Answer: (C) $2/r$**, (D)** $-2/r$

Q.10

$\triangle ABC$, $\angle ACB = \pi/6$, $a = x^2+x+1, b = x^2-1, c = 2x+1$. Find x.

Testing $x = 1+\sqrt{3}$: $a = 6+3\sqrt{3}, b = c = 3+2\sqrt{3}$ (isosceles), and direct substitution into the cosine rule $c^2 = a^2+b^2-2ab\cos(\pi/6)$ confirms equality exactly.

Answer: (B) $1+\sqrt{3}$

Q.11

z_1, z_2 distinct, $z = (1-t)z_1 + tz_2$, $0 < t < 1$. Which hold?

Since z lies strictly between z_1, z_2: distances add ($|z - z_1| + |z - z_2| = |z_1 - z_2|$, true); z, z_1, z_2 are collinear (the determinant condition holds); $z - z_1 = t(z_2 - z_1)$ is a positive multiple of $z_2 - z_1$ (same argument as $z_2 - z_1$). However $z - z_2 = -(1-t)(z_2 - z_1)$ has the *opposite* argument to $z - z_1$, not the same.

Answer: (A), (C), (D)

Q.12

$f(x) = \ln x + \int_0^x \sqrt{1+\sin t}\, dt$ on $(0, \infty)$. Which hold?

By FTC, $f'(x) = 1/x + \sqrt{1+\sin x}$, continuous everywhere but not differentiable at the isolated points where $\sin x = -1$ (square-root cusp) – matching (B). Since $f(x)$ grows essentially linearly while $f'(x)$ stays bounded, eventually $|f'(x)| < |f(x)|$ – (C) true; f is unbounded, so (D) is false.

Answer: (B), (C)

Q.13

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

This is the classical Dalzell integral, whose closed form is $\frac{22}{7} - \pi$.

Answer: (A) $22/7 - \pi$

Q.14

[Paragraph] p odd prime, $T_p = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \{0, \dots, p-1\} \right\}$. Count $A \in T_p$ that are symmetric or skew-symmetric (or both) with $p \mid \det A$.

Symmetric ($b = c$) with $p \mid (a^2 - b^2)$: gives $a = b$ (p matrices) or $a + b = p$ ($p - 1$ matrices), total $2p - 1$. Skew-symmetric requires $a = 0, c = -b$, and $p \mid \det$ forces $b = 0$ too (just the zero matrix). Union (subtracting the shared zero matrix): $(2p-1) + 1 - 1 = 2p - 1$.

Answer: (D) $2p - 1$

Q.15

[Paragraph, same T_p] Count $A \in T_p$ with trace not divisible by p but $\det A$ divisible by p.

Trace $= 2a \not\equiv 0 \pmod p \Leftrightarrow a \neq 0$ ($p - 1$ choices). For each such a, $bc \equiv a^2 \pmod p$ (nonzero) has exactly $p - 1$ solutions (b, c). Total: $(p-1)^2$.

Answer: (C) $(p-1)^2$

Q.16

[Paragraph, same T_p] Count $A \in T_p$ with $\det A$ not divisible by p.

Total matrices $= p^3$. Those with $p \mid \det A$: $(2p-1)$ from $a=0$ plus $(p-1)^2$ from $a \neq 0$, totalling p^2. So the complement has $p^3 - p^2$.
Answer: (D) $p^3 - p^2$

Q.17

[Paragraph] Circle $x^2+y^2-8x=0$ and hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ intersect at A, B. Common tangent with positive slope.

Circle: centre $(4,0)$, radius 4. Setting the tangent-to-hyperbola line $y = mx \pm \sqrt{9m^2-4}$ at distance 4 from $(4,0)$ leads to $495m^4 + 104m^2 - 400 = 0$, giving $m^2 = 4/5$, i.e. $m = 2/\sqrt{5}$, and the line $2x - \sqrt{5}y + 4 = 0$.
Answer: (B) $2x - \sqrt{5}y + 4 = 0$

Q.18

[Paragraph, same curves] Equation of the circle with AB as diameter.

Substituting the hyperbola relation into the circle equation gives $13x^2 - 72x - 36 = 0$, with only $x = 6$ valid (the other root fails $|x| \geq 3$). At $x = 6$, $y = \pm 2\sqrt{3}$, so $A = (6, 2\sqrt{3}), B = (6, -2\sqrt{3})$; the circle on AB as diameter is $x^2 + y^2 - 12x + 24 = 0$.
Answer: (A) $x^2 + y^2 - 12x + 24 = 0$

Q.19

$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\theta \neq n\pi/5$, satisfying both $\tan\theta = \cot 5\theta$ and $\sin 2\theta = \cos 4\theta$. Number of solutions.

The first equation gives 6 candidates $\theta = \frac{\pi}{12} + \frac{k\pi}{6}$ $(k = -3, \ldots, 2)$. Checking each against the second equation (which reduces to $\theta = \frac{\pi}{12} + \frac{m\pi}{3}$ or $\theta = -\frac{\pi}{4} + m\pi$) leaves exactly $\theta = \frac{\pi}{12}, -\frac{\pi}{4}, \frac{5\pi}{12}$ – none coinciding with an excluded $n\pi/5$.
Answer: 3

Q.20

Maximum value of $\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$.

The denominator simplifies to $3 + \frac{3}{2}\sin 2\theta + 2\cos 2\theta$, whose oscillating part has amplitude $\sqrt{(3/2)^2 + 2^2} = 5/2$. Minimum denominator $= 3 - 5/2 = 1/2$, giving maximum reciprocal $= 2$.
Answer: 2

Q.21

$\vec{a} = \frac{\hat{i}-2\hat{j}}{\sqrt{5}}$, $\vec{b} = \frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$. *Find* $(2\vec{a}+\vec{b}) \cdot [(\vec{a}\times\vec{b}) \times (\vec{a} - 2\vec{b})]$.

Both are unit vectors with $\vec{a}\cdot\vec{b} = 0$. Using $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$ with $\vec{c} = \vec{a} - 2\vec{b}$ gives $(\vec{a}\times\vec{b}) \times (\vec{a} - 2\vec{b}) = \vec{b} + 2\vec{a}$. Then $(2\vec{a}+\vec{b})\cdot(2\vec{a}+\vec{b}) = |2\vec{a}+\vec{b}|^2 = 4 + 0 + 1 = 5$.
Answer: 5

Q.22

Line $2x + y = 1$ tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, passing through the intersection of the nearer directrix and the x-axis. Find e.

Tangency gives $1 = 4a^2 - b^2$; the directrix condition gives $a/e = 1/2$, i.e. $e = 2a$. Combined with $b^2 = a^2(e^2 - 1)$, this reduces to $e^4 - 5e^2 + 4 = 0$, giving $e^2 = 4$ (rejecting $e^2 = 1$).
Answer: 2

Q.23

Distance between plane $Ax - 2y + z = d$ and the plane containing the given pair of lines is $\sqrt{6}$. Find $|d|$.

The plane containing the lines is $x - 2y + z = 0$ (matching $A = 1$). The distance between the parallel planes is $|d|/\sqrt{6} = \sqrt{6} \Rightarrow |d| = 6$.
Answer: 6

Q.24

$f(x) = x - [x]$ if $[x]$ odd, $1 + [x] - x$ if $[x]$ even, on $[-10, 10]$. Find $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x \, dx$.

$f(x) \cos \pi x$ has period 2; over one period, $\int_0^2 f(x) \cos \pi x \, dx = 4/\pi^2$ (computing the two triangular pieces separately). Over 10 periods: $40/\pi^2$, so the requested value is $\frac{\pi^2}{10} \cdot \frac{40}{\pi^2} = 4$.
Answer: 4

Q.25

$\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Number of distinct z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$.

Expanding the determinant (using $1 + \omega + \omega^2 = 0, \omega^3 = 1$) simplifies remarkably to $z^3 = 0$, giving the single distinct root $z = 0$.
Answer: 1