QuestIQ Academy

JEE Advanced 2014 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = comprehension-based single correct; Section 3 = matching-list.

Q.1

$\dfrac{dy}{dx} + \dfrac{xy}{x^2-1} = \dfrac{x^4+2x}{\sqrt{1-x^2}}$ *on* $(-1,1)$, $f(0)=0$. *Find* $\displaystyle\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\,dx$.

The integrating factor is $\exp\int \frac{x}{x^2-1}dx = \sqrt{1-x^2}$. So $\dfrac{d}{dx}[y\sqrt{1-x^2}] = x^4+2x$, giving $y\sqrt{1-x^2} = \frac{x^5}{5} + x^2$ (using $f(0)=0$), i.e. $f(x) = \dfrac{x^5/5+x^2}{\sqrt{1-x^2}}$. The x^5 term is odd (integrates to 0 over the symmetric interval); for the even part, $2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}dx$, substitute $x=\sin\theta$: $2\int_0^{\pi/3}\sin^2\theta\,d\theta = 2\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

Answer: (B) $\pi/3 - \sqrt{3}/4$

Q.2

$\displaystyle\int_{\pi/4}^{\pi/2} (2\csc x)^{17} dx$ *equals which of the given forms?*

Let $u = \ln(\csc x + \cot x)$, so $\csc x = \frac{1}{2}(e^u + e^{-u})$ and $du = -\csc x\,dx$. At $x=\pi/2$, $u=0$; at $x=\pi/4$, $u=\ln(1+\sqrt{2})$. Substituting: $(2\csc x)^{17}dx = (e^u+e^{-u})^{17}\cdot\dfrac{-2\,du}{e^u+e^{-u}} = -2(e^u+e^{-u})^{16}du$, and reversing the (decreasing) limits gives $2\int_0^{\ln(1+\sqrt{2})}(e^u+e^{-u})^{16}du$.

Answer: (A)

Q.3

Coefficient of x^{11} *in* $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$.

Need $2a+3b+4c=11$ with $0\le a\le 4, 0\le b\le 7, 0\le c\le 12$: solutions are $(a,b,c) = (4,1,0),(1,3,0),(2,1,1),(0,1,2)$. Coefficient $= \sum\binom{4}{a}\binom{7}{b}\binom{12}{c} = 1{\cdot}7{\cdot}1 + 4{\cdot}35{\cdot}1 + 6{\cdot}7{\cdot}12 + 1{\cdot}7{\cdot}66 = 7+140+504+462 = 1113$.

Answer: (C)

Q.4

$f:[0,2]\to\mathbb{R}$ continuous, differentiable on $(0,2)$, $f(0)=1$. $F(x)=\int_0^{x^2} f(\sqrt{t})\,dt$. If $F'(x)=f'(x)$ for all $x\in(0,2)$, find $F(2)$.

By the FTC and chain rule, $F'(x)=2x\,f(x)$. Setting $2xf(x)=f'(x)$ gives $\frac{f'(x)}{f(x)}=2x$, so $\ln f(x)=x^2+C$; using $f(0)=1$, $C=0$, so $f(x)=e^{x^2}$. Then $F(2)=\int_0^4 f(\sqrt{t})dt=\int_0^4 e^t dt=e^4-1$.
Answer: (B)

Q.5

Common tangents to $x^2+y^2=2$ and $y^2=8x$ touch the circle at P,Q and parabola at R,S. Area of quadrilateral $PQRS$.

Tangent to $y^2=8x$ ($a=2$): $y=mx+2/m$; tangent to the circle (radius $\sqrt{2}$) requires $4/m^2=2(m^2+1)\Rightarrow m^4+m^2-2=0\Rightarrow m^2=1$. For $m=1$: circle touches at $(-1,1)$, parabola at $(2,4)$. For $m=-1$: circle touches at $(-1,-1)$, parabola at $(2,-4)$. This is an isosceles trapezoid with parallel sides of length 2 (at $x=-1$) and 8 (at $x=2$), height 3: area $=\frac{1}{2}(2+8)(3)=15$.
Answer: (D)

Q.6

$\sin x+2\sin 2x-\sin 3x=3$ *for $x\in(0,\pi)$. Number of solutions.*

Using $\sin x-\sin 3x=-2\cos 2x\sin x$, the LHS simplifies to $2\sin x(2\cos x-\cos 2x)$. Writing $c=\cos x$: $2c-\cos 2x=-2(c-\frac{1}{2})^2+\frac{3}{2}\le\frac{3}{2}$. So LHS$\le 2\sin x\cdot\frac{3}{2}=3\sin x\le 3$, with equality requiring $\cos x=\frac{1}{2}$ (i.e. $x=\pi/3$) AND $\sin x=1$ (i.e. $x=\pi/2$) simultaneously – impossible. So the bound 3 is never attained.
Answer: (D)

Q.7

Triangle: sum of two sides $=x$, product $=y$; $x^2-c^2=y$ ($c=$ third side). Find (inradius)/(circumradius).

From $x^2-c^2=y$ with $p+q=x, pq=y$: $p^2+pq+q^2=c^2$; comparing to the law of cosines $c^2=p^2+q^2-2pq\cos C$ gives $\cos C=-\frac{1}{2}$, so $C=120°$, $\sin C=\frac{\sqrt{3}}{2}$. Then Area$=\frac{1}{2}pq\sin C=\frac{\sqrt{3}}{4}y$, $r=\text{Area}/s=\dfrac{\sqrt{3}y/4}{(x+c)/2}=\dfrac{\sqrt{3}y}{2(x+c)}$, and $R=\dfrac{c}{2\sin C}=\dfrac{c}{\sqrt{3}}$. So $r/R=\dfrac{3y}{2c(x+c)}$.
Answer: (B)

Q.8

6 cards/envelopes numbered 1–6; no card in its own-numbered envelope; card 1 always in envelope 2. Count the arrangements.

With card 1 fixed in envelope 2, the remaining cards $\{2,3,4,5,6\}$ fill envelopes $\{1,3,4,5,6\}$, with only cards $3,4,5,6$ carrying a forbidden envelope (card 2 has none, since envelope 2 is already

taken). By inclusion–exclusion on the 4 forbidden pairs: $5! - \binom{4}{1}4! + \binom{4}{2}3! - \binom{4}{3}2! + \binom{4}{4}1! = 120 - 96 + 36 - 8 + 1 = 53$.
Answer: (C)

Q.9

3 boys, 2 girls in a queue. P(boys ahead of every girl $\geq$ girls ahead of her $+ 1$).

Only the relative B/G pattern matters; there are $\binom{5}{2} = 10$ equally likely patterns. If the two girls are at positions $p < q$, the condition reduces to $p \geq 2$ and $q \geq 4$. Checking all 10 patterns, this holds for $(p, q) = (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)$ – 5 of the 10.
Answer: (A) 1/2

Q.10

$p(x) = 0$ (real quadratic) has purely imaginary roots $\pm ik$. What roots does $p(p(x)) = 0$ have?

$p(x) = a(x^2 + k^2)$. Solving $p(x) = \pm ik$ gives $x^2 = \dfrac{k(i \mp ak)}{a}$ wait – concretely $x^2 = \dfrac{ik}{a} - k^2$, a complex number with nonzero real part $(-k^2)$ and nonzero imaginary part (k/a). Since x^2 is neither real nor purely imaginary, its square roots x can be neither purely real (which would force $x^2 \geq 0$ real) nor purely imaginary (which would force $x^2 \leq 0$ real).
Answer: (D)

Q.11

$P(at^2, 2at), R(ar^2, 2ar)$ on $y^2 = 4ax$; PQ is a focal chord (so Q has parameter $-1/t$); $QR \parallel PK$, $K = (2a, 0)$. Find r.

Slope of $PK = \dfrac{2t}{t^2 - 2}$. Slope of $QR = \dfrac{2}{r - 1/t}$ (after simplifying using $Q = (a/t^2, -2a/t)$).
Equating: $t^2 - 2 = t(r - 1/t) = tr - 1 \Rightarrow r = \dfrac{t^2 - 1}{t}$.
Answer: (D)

Q.12

Continuing Q.11 with $st = 1$: the tangent at P and normal at S meet at a point with what ordinate?

Tangent at P: $ty = x + at^2$. Normal at S (parameter $s = 1/t$): $y = -sx + 2as + as^3 = -\frac{x}{t} + \frac{2a}{t} + \frac{a}{t^3}$. Substituting $x = ty - at^2$ from the tangent equation and solving: $2y = at + \frac{2a}{t} + \frac{a}{t^3} = \dfrac{a(t^2 + 1)^2}{2t^3} \cdot 2$, giving $y = \dfrac{a(t^2 + 1)^2}{2t^3}$.
Answer: (B)

Q.13

$g(a) = \lim_{h \to 0^+} \int_h^{1-h} t^{-a}(1 - t)^{a-1} dt$. Find $g(1/2)$.

This is the Beta function $B(1-a,a) = \Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin \pi a}$ (reflection formula). At $a = 1/2$: $g(1/2) = \pi/\sin(\pi/2) = \pi$.
Answer: (A)

Q.14

Continuing Q.13: find $g'(1/2)$.

$g(a) = \pi \csc(\pi a)$, so $g'(a) = -\pi^2 \csc(\pi a)\cot(\pi a)$. At $a = 1/2$: $\cot(\pi/2) = 0$, so $g'(1/2) = 0$.
Answer: (D)

Q.15

Boxes with cards $\{1,2,3\}, \{1,...,5\}, \{1,...,7\}$; draw x_1, x_2, x_3. Find $P(x_1 + x_2 + x_3$ odd).

Each box has (odd,even) counts $(2,1), (3,2), (4,3)$ out of $(3,5,7)$. Summing over the parity combinations giving an odd total: $\dfrac{2\cdot3\cdot4 + 2\cdot2\cdot3 + 1\cdot3\cdot3 + 1\cdot2\cdot4}{105} = \dfrac{24+12+9+8}{105} = \dfrac{53}{105}$.
Answer: (B)

Q.16

Continuing Q.15: find $P(x_1, x_2, x_3$ in AP).

Need $x_1 + x_3 = 2x_2$ with $x_1 \in [1,3], x_2 \in [1,5], x_3 \in [1,7]$. Counting valid (x_1, x_3) pairs for each $x_2 = 1, \ldots, 5$ gives $1, 3, 3, 3, 1$ respectively, totalling 11 out of 105.
Answer: (C)

Q.17

$z_k = \cos(2k\pi/10) + i\sin(2k\pi/10)$, $k = 1, \ldots, 9$ (nontrivial 10th roots of unity). Match:
P. For each z_k there is z_j with $z_k z_j = 1$.
Q. There is k for which $z_1 z = z_k$ has no solution $z \in \mathbb{C}$.
R. $\dfrac{|1-z_1|\cdots|1-z_9|}{10}$ equals.
S. $1 - \sum_{k=1}^{9} \cos(2k\pi/10)$ equals.

Since $|z_k| = 1$, $z_k^{-1} = \bar{z}_k$ is also a nontrivial 10th root of unity, so P is always true. Since $z_1 \neq 0$, $z = z_k/z_1$ always solves the equation, so Q is always false. Since $x^{10} - 1 = (x-1)\prod_{k=1}^{9}(x - z_k)$, evaluating $\frac{x^{10}-1}{x-1} = 1 + x + \cdots + x^9$ at $x = 1$ gives $\prod(1 - z_k) = 10$, so R $= 10/10 = 1$. Since $\sum_{k=0}^{9} z_k = 0$, $\sum_{k=1}^{9} z_k = -1$, so $\sum \cos(2k\pi/10) = -1$ and S $= 1 - (-1) = 2$.
Answer: (A) P→True, Q→False, R→ 1, S→ 2

Q.18

Match:
P. Number of polynomials $f(x) = ax^2 + bx$ ($f(0) = 0$), nonnegative integer coefficients, with $\int_0^1 f(x)dx = 1$.

Q. Number of maxima of $\sin(x^2)+\cos(x^2)$ *on* $[-\sqrt{13},\sqrt{13}]$.
R. $\int_{-2}^{2}\frac{3x^2}{1+e^x}dx$.
S. Ratio of $\int_{-1/2}^{1/2}\cos 2x\log\frac{1+x}{1-x}\,dx$ *to* $\int_0^{1/2}$ *of the same.*

P: $\int_0^1(ax^2+bx)dx=\frac{a}{3}+\frac{b}{2}=1\Rightarrow 2a+3b=6$; nonnegative integer solutions $(3,0),(0,2)$ – 2 solutions. Q: maxima occur where $x^2=\frac{\pi}{4}+2n\pi\leq 13$; $n=0,1$ each give 2 points ($\pm$), totalling 4. R: since $3x^2$ is even, the symmetry trick gives $\frac{1}{2}\int_{-2}^{2}3x^2dx=\frac{1}{2}\cdot 16=8$. S: the integrand is odd (even $\times$ odd), so the numerator (symmetric-interval integral) is 0.
Answer: (D) P$\rightarrow$ 2, Q$\rightarrow$ 4, R$\rightarrow$ 8, S$\rightarrow$ 0

Q.19

Match:
P. $y(x)=\cos(3\cos^{-1}x)$*: find* $\frac{1}{y}\{(x^2-1)y''+xy'\}$.
Q. Regular n-gon, vertices $A_1,\ldots,A_n$ *centred at origin, position vectors* $\vec{a}_k$*. If* $|\sum_{k=1}^{n-1}\vec{a}_k\times\vec{a}_{k+1}|=|\sum_{k=1}^{n-1}\vec{a}_k\cdot\vec{a}_{k+1}|$*, find the minimum n.*
R. Normal at $P(h,1)$ *on* $\frac{x^2}{6}+\frac{y^2}{3}=1$ *is* $\perp$ *to* $x+y=8$*: find h.*
S. Positive solutions of $\tan^{-1}\frac{1}{2x+1}+\tan^{-1}\frac{1}{4x+1}=\tan^{-1}\frac{2}{x^2}$.

P: $y=T_3(x)$ satisfies the Chebyshev equation $(1-x^2)y''-xy'+9y=0$, so $\frac{1}{y}\{(x^2-1)y''+xy'\}=9$. Q: with all vertices on a common circle, $|\sum\vec{a}_k\times\vec{a}_{k+1}|=(n-1)R^2\sin(2\pi/n)$ and $|\sum\vec{a}_k\cdot\vec{a}_{k+1}|=(n-1)R^2|\cos(2\pi/n)|$; equality needs $\tan(2\pi/n)=1$, smallest at $n=8$. R: the normal $\frac{6x}{h}-3y=3$ has slope $2/h$; perpendicular to slope -1 gives $h=2$. S: combining the arctangents gives $3x^2-7x-6=0\Rightarrow x=3$ or $x=-\frac{2}{3}$; only $x=3$ is positive.
Answer: (A) P$\rightarrow$ 9, Q$\rightarrow$ 8, R$\rightarrow$ 2, S$\rightarrow$ 1

Q.20

$f_1(x)=|x|\,(x<0)$, $e^x\,(x\geq 0)$*;* $f_2(x)=x^2$ *on* $[0,\infty)$*;* $f_3(x)=\sin x\,(x<0)$, $x\,(x\geq 0)$*;* $f_4(x)=f_2(f_1(x))\,(x<0)$, $f_2(f_1(x))-1\,(x\geq 0)$*. Match each to: onto but not one-one / neither continuous nor one-one / differentiable but not one-one / continuous and one-one.*

$f_4(x)=x^2\,(x<0)$, $e^{2x}-1\,(x\geq 0)$: both branches' ranges overlap on $(0,\infty)$ (not one-one) while together covering all of $[0,\infty)$ (onto) – "onto but not one-one". f_3 is smooth at 0 (both one-sided derivatives equal 1) but $\sin x$ is not injective on $(-\infty,0)$ – "differentiable but not one-one". $f_2\circ f_1(x)=x^2\,(x<0)$, $e^{2x}\,(x\geq 0)$ jumps from 0 to 1 at $x=0$ (discontinuous) and its branches overlap in range – "neither continuous nor one-one". f_2 restricted to $[0,\infty)$ is strictly increasing and continuous – "continuous and one-one".
Answer: (D) f_4 $\rightarrow$onto but not one-one, f_3 $\rightarrow$differentiable but not one-one, $f_2\circ f_1$ $\rightarrow$neither continuous nor one-one, f_2 $\rightarrow$continuous and one-one