QuestIQ Academy

JEE Advanced 2015 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single-digit integer; Section 2 = one or more correct; Section 3 = paragraph-based, one or more correct.

Q.1

$\alpha_k = \cos(k\pi/7) + i\sin(k\pi/7)$. *Find* $\dfrac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$.

Since $\alpha_k = e^{ik\pi/7}$, $|\alpha_{k+1} - \alpha_k| = |e^{ik\pi/7}||e^{i\pi/7} - 1| = |e^{i\pi/7} - 1|$, a constant independent of k. So the numerator is 12 times this constant and the denominator is 3 times the same constant.

Answer: 4

Q.2

All terms of an AP are natural numbers; $S_7 : S_{11} = 6 : 11$*; the* 7*th term lies between* 130 *and* 140*. Find the common difference.*

$\dfrac{7(2a+6d)}{11(2a+10d)} = \dfrac{6}{11} \Rightarrow 7(2a+6d) = 6(2a+10d) \Rightarrow a = 9d$. The 7th term is $a + 6d = 15d$; requiring $130 < 15d < 140$ gives $d = 9$ (with $a = 81$, 7th term $= 135$).

Answer: 9

Q.3

Coefficient of x^9 *in* $(1+x)(1+x^2)(1+x^3)\cdots(1+x^{100})$.

This equals the number of ways to write 9 as a sum of distinct positive integers (each ≤ 100): $\{9\}$; $\{1,8\},\{2,7\},\{3,6\},\{4,5\}$; $\{1,2,6\},\{1,3,5\},\{2,3,4\}$ – a total of $1+4+3 = 8$ partitions.

Answer: 8

Q.4

Foci of $\frac{x^2}{9} + \frac{y^2}{5} = 1$ *are* $(f_1, 0), (f_2, 0)$ $(f_1 > 0 > f_2)$*.* P_1, P_2 *are parabolas, vertex* $(0,0)$*, with foci* $(f_1, 0)$ *and* $(2f_2, 0)$*.* T_1 *is tangent to* P_1 *through* $(2f_2, 0)$*;* T_2 *is tangent to* P_2 *through* $(f_1, 0)$*, with slopes* m_1, m_2*. Find* $1/m_1^2 + m_2^2$.

$c^2 = 9 - 5 = 4$, so $f_1 = 2, f_2 = -2$; $2f_2 = -4$. $P_1 : y^2 = 8x$ (focus $(2,0)$); its tangent $y = mx + 2/m$ through $(-4,0)$ gives $4m = 2/m \Rightarrow m_1^2 = \frac{1}{2}$. $P_2 : y^2 = -16x$ (focus $(-4,0)$); its tangent $y = mx - 4/m$ through $(2,0)$ gives $2m = 4/m \Rightarrow m_2^2 = 2$. So $1/m_1^2 + m_2^2 = 2 + 2 = 4$.
Answer: 4

Q.5

m, n integers > 1. $\lim_{a\to 0} \dfrac{e^{\cos(a^n)} - e}{a^m} = -\dfrac{e}{2}$. Find m/n.

Using $\cos(a^n) \approx 1 - \frac{1}{2}a^{2n}$, $e^{\cos(a^n)} - e \approx e\big(e^{-a^{2n}/2} - 1\big) \approx -\frac{e}{2}a^{2n}$. For the limit to be a nonzero finite value, $m = 2n$, so $m/n = 2$.
Answer: 2

Q.6

$\alpha = \displaystyle\int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\frac{12+9x^2}{1+x^2}dx$ (principal values). Find $\left(\log_e|1+\alpha| - \dfrac{3\pi}{4}\right)$.

Since $\dfrac{d}{dx}(9x + 3\tan^{-1}x) = 9 + \dfrac{3}{1+x^2} = \dfrac{12+9x^2}{1+x^2}$, the integrand is exactly $\dfrac{d}{dx}\big[e^{9x+3\tan^{-1}x}\big]$. So $\alpha = \big[e^{9x+3\tan^{-1}x}\big]_0^1 = e^{9+3\pi/4} - 1$, hence $1 + \alpha = e^{9+3\pi/4}$ and $\log_e|1+\alpha| - \frac{3\pi}{4} = 9$.
Answer: 9

Q.7

$f : \mathbb{R} \to \mathbb{R}$ continuous odd, vanishing at exactly one point, $f(1) = \frac{1}{2}$. $F(x) = \int_{-1}^x f(t)dt$, $G(x) = \int_{-1}^x t|f(f(t))|dt$. If $\lim_{x\to 1} F(x)/G(x) = 1/14$, find $f(1/2)$.

Since f is odd with a unique zero, $F(1) = \int_{-1}^1 f = 0$; also $t|f(f(t))|$ is odd in t (as $f(f(t))$ is an odd composition of an odd function), so $G(1) = 0$ too. By L'Hopital, the limit equals $\dfrac{f(1)}{1\cdot|f(f(1))|} = \dfrac{1/2}{|f(1/2)|} = \dfrac{1}{14}$, so $|f(1/2)| = 7$.
Answer: 7

Q.8

$\vec{p}, \vec{q}, \vec{r}$ non-coplanar. Components of $\vec{s}$ along $\vec{p}, \vec{q}, \vec{r}$ are $4, 3, 5$. Components of $\vec{s}$ along $(-\vec{p} + \vec{q} + \vec{r}), (\vec{p} - \vec{q} + \vec{r}), (-\vec{p} - \vec{q} + \vec{r})$ are x, y, z. Find $2x + y + z$.

With $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r} = x(-\vec{p} + \vec{q} + \vec{r}) + y(\vec{p} - \vec{q} + \vec{r}) + z(-\vec{p} - \vec{q} + \vec{r})$, matching coefficients gives $-x + y - z = 4$, $x - y - z = 3$, $x + y + z = 5$. Solving: $z = -\frac{7}{2}$, $y = \frac{9}{2}$, $x = 4$, so $2x + y + z = 8 + \frac{9}{2} - \frac{7}{2} = 9$.
Answer: 9

Q.9

$S = \{\alpha \neq 0 : \alpha x^2 - x + \alpha = 0$ has distinct real roots x_1, x_2 with $|x_1 - x_2| < 1\}$. Which intervals are subset(s) of S?

Discriminant $1-4\alpha^2 > 0 \Rightarrow |\alpha| < \frac{1}{2}$. Root difference $|x_1 - x_2| = \sqrt{1-4\alpha^2}/|\alpha| < 1 \Rightarrow 1-4\alpha^2 < \alpha^2 \Rightarrow \alpha^2 > \frac{1}{5}$. So $S = \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$.

Answer: (A), (D)

Q.10

$\alpha = 3\sin^{-1}(6/11)$, $\beta = 3\cos^{-1}(4/9)$ (principal values). Which is (are) true regarding $\cos\beta, \sin\beta, \cos(\alpha+\beta), \cos\alpha$?

Let $\theta = \sin^{-1}(6/11)$: since $\sin\theta = 6/11$, $\cos\theta = \sqrt{85}/11$, and $\cos 3\theta = \cos\theta(4\cos^2\theta - 3) = -23\sqrt{85}/1331 < 0$, so $\cos\alpha < 0$ (true). $\sin 3\theta = \sin\theta(3 - 4\sin^2\theta) = 1314/1331 > 0$, so $\sin\alpha > 0$. Let $\varphi = \cos^{-1}(4/9)$: $\sin\varphi = \sqrt{65}/9$, and $\sin 3\varphi = \sin\varphi(3 - 4\sin^2\varphi) = -17\sqrt{65}/729 < 0$, so $\sin\beta < 0$ (true); $\cos 3\varphi = \cos\varphi(4\cos^2\varphi - 3) = -716/729 < 0$, so $\cos\beta < 0$, meaning $\cos\beta > 0$ is false. Finally $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$: both terms are positive (negative×negative, and −(positive×negative)), so $\cos(\alpha+\beta) > 0$ (true).

Answer: (B), (C), (D)

Q.11

E_1, E_2 ellipses centred at the origin, major axes along x, y respectively. $S : x^2 + (y-1)^2 = 2$. The line $x + y = 3$ touches S, E_1, E_2 at P, Q, R, with $PQ = PR = \frac{2\sqrt{2}}{3}$. Find relations among the eccentricities e_1, e_2.

$P = (1, 2)$ (tangent point of the line to S). The two points on the line at distance $\frac{2\sqrt{2}}{3}$ from P are $\left(\frac{5}{3}, \frac{4}{3}\right)$ and $\left(\frac{1}{3}, \frac{8}{3}\right)$. Writing $E_1 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ with tangent $\frac{x_0}{A^2}x + \frac{y_0}{B^2}y = 1$ matching $\frac{x}{3} + \frac{y}{3} = 1$: taking $Q = \left(\frac{5}{3}, \frac{4}{3}\right)$ gives $A^2 = 5, B^2 = 4$ (valid since $A > B$), so $e_1^2 = 1 - \frac{4}{5} = \frac{1}{5}$. Then $R = \left(\frac{1}{3}, \frac{8}{3}\right)$ gives E_2's $C^2 = 1, D^2 = 8$ ($D > C$, major axis y), so $e_2^2 = 1 - \frac{1}{8} = \frac{7}{8}$. Then $e_1^2 + e_2^2 = \frac{1}{5} + \frac{7}{8} = \frac{43}{40}$ and $e_1 e_2 = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{7}}{2\sqrt{10}}$.

Answer: (A), (B)

Q.12

Hyperbola $H : x^2 - y^2 = 1$, circle S centred $N(x_2, 0)$, tangent at $P(x_1, y_1)$ ($x_1 > 1, y_1 > 0$). Common tangent meets the x-axis at M. (l, m) = centroid of $\triangle PMN$. Which is (are) true?

Tangent to H at P: $xx_1 - yy_1 = 1$, so $M = (1/x_1, 0)$. Since the radius $NP \perp$ tangent, matching slopes gives $N = (2x_1, 0)$. Centroid: $l = x_1 + \frac{1}{3x_1}$, $m = y_1/3 = \sqrt{x_1^2 - 1}/3$. Then $\dfrac{dl}{dx_1} = 1 - \dfrac{1}{3x_1^2}$ (true) and $\dfrac{dm}{dx_1} = \dfrac{x_1}{3\sqrt{x_1^2 - 1}}$ (true); also directly $\dfrac{dm}{dy_1} = \dfrac{1}{3}$ (true).

Answer: (A), (B), (D)

Q.13

Find (a, L) satisfying $\dfrac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L$.

If $g(t) = \sin^6 at + \cos^4 at$ has period π (true whenever $a = 2$ or $a = 4$, since then at has period π or a divisor of it), splitting $[0, 4\pi]$ into four copies of $[0, \pi]$ shifted by $k\pi$ gives $\int_0^{4\pi} e^t g\,dt = \left(\sum_{k=0}^{3} e^{k\pi}\right) \int_0^{\pi} e^t g\,dt$, so $L = \dfrac{e^{4\pi} - 1}{e^{\pi} - 1}$ for both $a = 2$ and $a = 4$.

Answer: (A), (C)

Q.14

$f, g : [-1, 2] \to \mathbb{R}$ twice differentiable on $(-1, 2)$, with $f(-1) = 3, f(0) = 6, f(2) = 0$ and $g(-1) = 0, g(0) = 1, g(2) = -1$. $(f - 3g)''$ never vanishes on $(-1, 0)$ or $(0, 2)$. Which is (are) true of $f'(x) - 3g'(x) = 0$?

Let $h = f - 3g$: $h(-1) = 3, h(0) = 3, h(2) = 3$ – equal at all three points. By Rolle's theorem, $h' = 0$ has a root in $(-1, 0)$ and in $(0, 2)$; since $h'' \neq 0$ on each interval, h' is strictly monotonic there, so exactly one root in each.

Answer: (B), (C)

Q.15

$f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$. Which is (are) true?

Since $7\tan^6 x \sec^2 x = \frac{d}{dx}\tan^7 x$ and $3\tan^2 x \sec^2 x = \frac{d}{dx}\tan^3 x$, $f(x) = \frac{d}{dx}\big[\tan^7 x - \tan^3 x\big]$. So $\int_0^{\pi/4} f\,dx = \big[\tan^7 x - \tan^3 x\big]_0^{\pi/4} = 0$ (true). By parts, $\int_0^{\pi/4} xf\,dx = \big[x(\tan^7 x - \tan^3 x)\big]_0^{\pi/4} - \int_0^{\pi/4}(\tan^7 x - \tan^3 x)dx = 0 - \left(-\frac{1}{12}\right) = \frac{1}{12}$ (using the standard reduction formula for $\int \tan^n x\,dx$), matching (A) and excluding (C),(D).

Answer: (A), (B)

Q.16

$f'(x) = \dfrac{192x^3}{2 + \sin^4 \pi x}$, $f(1/2) = 0$. If $m \leq \int_{1/2}^{1} f(x)dx \leq M$, find possible (m, M).

Since $\sin^4 \pi x \in [0, 1]$, $64x^3 \leq f'(x) \leq 96x^3$. Using $f(x) = \int_{1/2}^{x} f'(t)dt$ and swapping the order of integration, $\int_{1/2}^{1} f(x)dx = \int_{1/2}^{1}(1 - t)f'(t)dt$, which lies between $64\int_{1/2}^{1}(1 - t)t^3 dt = \frac{13}{5}$ and $96\int_{1/2}^{1}(1 - t)t^3 dt = \frac{39}{10}$. Among the given options, only $[1, 12]$ contains this range.

Answer: (D)

Q.17

Box I has n_1 red, n_2 black balls; box II has n_3 red, n_4 black. A box is picked at random and a red ball drawn. If $P(\text{box II} \mid \text{red}) = \frac{1}{3}$, which values of (n_1, n_2, n_3, n_4) are possible?

With $p_1 = n_1/(n_1+n_2)$, $p_2 = n_3/(n_3+n_4)$, Bayes' theorem gives $\dfrac{p_2}{p_1+p_2} = \dfrac{1}{3} \Rightarrow p_1 = 2p_2$. Checking: $(3,3,5,15)$: $p_1 = \frac{1}{2}, p_2 = \frac{1}{4}$, $p_1 = 2p_2$ (true); $(3,6,10,50)$: $p_1 = \frac{1}{3}, p_2 = \frac{1}{6}$, $p_1 = 2p_2$ (true); the other two options fail this ratio.
Answer: (A), (B)

Q.18

A ball drawn from box I is transferred to box II. If the probability of then drawing a red ball from the (reduced) box I is $\frac{1}{3}$, which values of (n_1, n_2) are possible?

Averaging over which ball was removed, $P(\text{red from reduced box I}) = \dfrac{n_1(n_1-1)+n_2n_1}{(n_1+n_2)(n_1+n_2-1)} = \dfrac{n_1}{n_1+n_2}$ (a clean simplification). Setting this to $\frac{1}{3}$: $(10,20)$ gives $\frac{10}{30} = \frac{1}{3}$ (true); $(3,6)$ gives $\frac{3}{9} = \frac{1}{3}$ (true); the other two options fail.
Answer: (C), (D)

Q.19

$F : \mathbb{R} \to \mathbb{R}$ thrice differentiable, $F(1) = 0, F(3) = -4$, $F'(x) < 0$ on $(1/2, 3)$. $f(x) = xF(x)$. Which is (are) true?

$f'(x) = F(x) + xF'(x)$. At $x = 1$: $f'(1) = F(1) + F'(1) = F'(1) < 0$ (true, since $F' < 0$ there). Since F is strictly decreasing with $F(1) = 0$, $F(2) < 0$, so $f(2) = 2F(2) < 0$ (true). For $x \in (1,3)$, both $F(x) < 0$ (as F decreases past 0) and $F'(x) < 0$, so $f'(x) = F(x) + xF'(x) < 0$ throughout – never zero (true (C), false (D)).
Answer: (A), (B), (C)

Q.20

Continuing Q.19's setup: if $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$, which is (are) true?

Integrating by parts, $\int_1^3 x^2F'(x)dx = \left[x^2F(x)\right]_1^3 - 2\int_1^3 xF(x)dx = 9F(3) - 2\int_1^3 f(x)dx = -36 - 2\int_1^3 f(x)dx = -12$, giving $\int_1^3 f(x)dx = -12$. Similarly $\int_1^3 x^3F''(x)dx = \left[x^3F'(x)\right]_1^3 - 3\int_1^3 x^2F'(x)dx = 27F'(3) - F'(1) - 3(-12) = 40$, giving $27F'(3) - F'(1) = 4$. Using $f'(1) = F'(1)$ and $f'(3) = F(3) + 3F'(3) = -4 + 3F'(3)$ (so $F'(3) = (f'(3)+4)/3$), substitution gives $9f'(3) - f'(1) + 32 = 0$.
Answer: (C), (D)