QuestIQ Academy

JEE Advanced 2016 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = single-digit integer.

Q.1

$-\pi/6 < \theta < -\pi/12$. α_1, β_1 roots of $x^2 - 2x\sec\theta + 1 = 0$ ($\alpha_1 > \beta_1$); α_2, β_2 roots of $x^2 + 2x\tan\theta - 1 = 0$ ($\alpha_2 > \beta_2$). Find $\alpha_1 + \beta_2$.

Solving the quadratics: $\alpha_1, \beta_1 = \sec\theta \pm |\tan\theta|$ and $\alpha_2, \beta_2 = -\tan\theta \pm \sec\theta$ (using $\sec\theta > 0$ here). Since $\theta < 0$, $\tan\theta < 0$ so $|\tan\theta| = -\tan\theta$, giving $\alpha_1 = \sec\theta - \tan\theta$ and $\beta_2 = -\tan\theta - \sec\theta$. Sum: $\alpha_1 + \beta_2 = -2\tan\theta$.
Answer: (C) $-2\tan\theta$

Q.2

Debate club: 6 girls, 4 boys. Team of 4 (with a captain chosen from the 4) including at most one boy. Count the ways.

0 boys: $\binom{6}{4}$ teams $\times 4$ captain choices $= 15 \times 4 = 60$. 1 boy: $\binom{6}{3}\binom{4}{1}$ teams $\times 4$ captain choices $= 80 \times 4 = 320$. Total $= 60 + 320$.
Answer: (A) 380

Q.3

$S = \{x \in (-\pi, \pi) : x \neq 0, \pm\pi/2\}$. Sum of all distinct solutions of $\sqrt{3}\sec x + \csc x + 2(\tan x - \cot x) = 0$ in S.

Multiplying through by $\sin x \cos x$ gives $\sqrt{3}\sin x + \cos x = 2\cos 2x$, i.e. $2\sin(x + \pi/6) = 2\cos 2x = 2\sin(\pi/2 - 2x)$. Solving the two general-solution branches within $(-\pi, \pi)$ gives $x = \pi/9,\ 7\pi/9,\ -5\pi/9,\ -\pi/3$, none of which are excluded values. Their sum telescopes to 0.
Answer: (C) 0

Q.4

Plant T_1: 20% of output, T_2: 80%. $P(\text{defective} \mid T_1) = 10\,P(\text{defective} \mid T_2)$; overall 7% defective. Given a computer is NOT defective, find $P(T_2)$.

Let $P(D \mid T_2) = p$; then $0.2(10p)+0.8p = 0.07 \Rightarrow 2.8p = 0.07 \Rightarrow p = 0.025$, so $P(D \mid T_1) = 0.25$. $P(\text{not } D) = 0.2(0.75)+0.8(0.975) = 0.93$. By Bayes' theorem, $P(T_2 \mid \text{not } D) = \dfrac{0.8 \times 0.975}{0.93} = \dfrac{0.78}{0.93}$.
Answer: (C) 78/93

Q.5

Least $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \dfrac{1}{x} \geq 1$ for all $x > 0$.

For $\alpha > 0$, minimising $f(x) = 4\alpha x^2 + 1/x$ gives $f'(x) = 8\alpha x - 1/x^2 = 0 \Rightarrow x^3 = 1/(8\alpha)$, and at this point $f(x) = 3/(2x)$. Requiring the minimum to just reach 1 gives $x = 3/2$, hence $\alpha = 1/(8x^3) = 1/(8 \cdot 27/8) = 1/27$.
Answer: (C) 1/27

Q.6

Pyramid $OPQRS$ in the first octant: O origin, OP, OR along axes, base $OPQR$ a square with $OP = 3$; S is above the midpoint T of diagonal OQ with $TS = 3$.

$O = (0,0,0)$, $P = (3,0,0)$, $R = (0,3,0)$, $Q = (3,3,0)$, $T = (1.5,1.5,0)$, $S = (1.5,1.5,3)$. (A) $\cos(\angle QOS) = \frac{\vec{OQ}\cdot\vec{OS}}{|\vec{OQ}||\vec{OS}|} = \frac{9}{3\sqrt{2}\cdot 3\sqrt{6}/2} = \frac{1}{\sqrt{3}} \neq \frac{1}{2}$, so this is false. (B) Plane OQS has normal $\vec{OQ} \times \vec{OS} \parallel (1,-1,0)$, giving $x - y = 0$, true. (C) Distance from $P = (3,0,0)$ to $x - y = 0$ is $3/\sqrt{2}$, true. (D) Distance from O to line RS (through $R = (0,3,0)$, direction $(1.5,-1.5,3)$) works out to $\sqrt{15/2}$, true.
Answer: (B), (C), (D)

Q.7

$f : (0,\infty) \to \mathbb{R}$ differentiable, $f'(x) = 2 - f(x)/x$, $f(1) \neq 1$.

This is the linear ODE $f' + f/x = 2$, giving $(xf)' = 2x$, so $f(x) = x + C/x$ with $C \neq 0$ (since $f(1) = 1 + C \neq 1$). (A) $f'(x) = 1 - C/x^2$, so $f'(1/x) = 1 - Cx^2 \to 1$ as $x \to 0^+$, true. (B) $xf(1/x) = x(1/x + Cx) = 1 + Cx^2 \to 1 \neq 2$, false. (C) $x^2 f'(x) = x^2 - C \to -C \neq 0$, false. (D) Since $C \neq 0$, $f(x) \to \pm\infty$ as $x \to 0^+$, so $|f(x)| \leq 2$ fails near 0, false.
Answer: (A)

Q.8

$P = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{pmatrix}$, *$Q = [q_{ij}]$ with $PQ = kI$ $(k \neq 0)$, $q_{23} = -k/8$, $\det(Q) = k^2/2$.*

Since $PQ = kI$, $Q = k\,\text{adj}(P)/\det(P)$, and $\det(Q) = k^3/\det(P)$; combined with $\det Q = k^2/2$ this gives $\det P = 2k$. Direct expansion gives $\det P = 12\alpha + 20 = 2k$. Matching $q_{23} = (\text{adj}P)_{23}/2$ (since $Q = \text{adj}(P)/2$ once $\det P = 2k$) to $-(3\alpha + 4)/2 = -k/8$ and solving simultaneously gives $\alpha = -1, k = 4$ (so (A) is false, and $4\alpha - k + 8 = -4 - 4 + 8 = 0$, (B) true). Then $\det P = 8$, $\det(P\,\text{adj}Q) = \det P \cdot (\det Q)^2 = 8 \cdot 64 = 2^9$ (C true), while $\det(Q\,\text{adj}P) = \det Q \cdot (\det P)^2 = 8 \cdot 64 = 2^9 \neq 2^{13}$ (D false).

Answer: (B), (C)

Q.9

$\triangle XYZ$: $2s = x + y + z$, $\dfrac{s-x}{4} = \dfrac{s-y}{3} = \dfrac{s-z}{2}$, *incircle area* $= 8\pi/3$.

Setting the common ratio to t gives $s = 9t$ and sides $x = 5t, y = 6t, z = 7t$. From $r^2 = 8/3$ and $r = \text{Area}/s$ together with Heron's formula Area $= 6\sqrt{6}\,t^2$, solving gives $t = 1$: sides $5, 6, 7$, Area $= 6\sqrt{6}$ (A true). Circumradius $R = \dfrac{xyz}{4\text{Area}} = \dfrac{210}{24\sqrt{6}} = \dfrac{35\sqrt{6}}{24} \neq \dfrac{35\sqrt{6}}{6}$ (B false). Using $r = 4R\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$: $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \dfrac{r}{4R} = \dfrac{4}{35}$ (C true). With $\cos Z = \frac{x^2+y^2-z^2}{2xy} = \frac{1}{5}$: $\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{1+\cos Z}{2} = \frac{3}{5}$ (D true).
Answer: (A), (C), (D)

Q.10

$(x^2 + xy + 4x + 2y + 4)\dfrac{dy}{dx} - y^2 = 0$, $x > 0$, *through* $(1, 3)$.

The denominator factors as $(x+2)(x+y+2)$. Substituting $u = x+2$, $y = vu$ reduces the equation to a separable form, integrating to $\ln|y| + \dfrac{y}{x+2} = 1 + \ln 3$ (using the initial condition). Substituting $y = x+2$ gives $\ln(x+2) + 1 = 1 + \ln 3 \Rightarrow x = 1$ – exactly one intersection (A true, B false). Substituting $y = (x+2)^2$ gives $2\ln(x+2) + (x+2) = 1 + \ln 3$, whose LHS already exceeds the RHS at $x = 0$ and keeps increasing – no solution for $x > 0$ (C false). Substituting $y = (x+3)^2$ similarly never reaches $1 + \ln 3$ for $x > 0$, so the curve does not intersect it (D true).
Answer: (A), (D)

Q.11

$f(x) = x^3 + 3x + 2$, $g(f(x)) = x$, $h(g(g(x))) = x$ *for all* x.

Since f is strictly increasing (as $f'(x) = 3x^2 + 3 > 0$), $g = f^{-1}$, and $h = (g \circ g)^{-1} = f \circ f$. (A) $g(2) = 0$ (since $f(0) = 2$), $f'(0) = 3$, so $g'(2) = 1/f'(0) = 1/3 \neq 1/15$, false. (B) $h'(1) = f'(f(1))f'(1) = f'(6)\cdot 6 = 111 \times 6 = 666$, true. (C) $h(0) = f(f(0)) = f(2) = 16$, true. (D) $g(3)$ solves $f(x) = 3$; $h(g(3)) = f(f(g(3))) = f(3) = 38 \neq 36$, false.
Answer: (B), (C)

Q.12

$C_1 : x^2 + y^2 = 3$ *meets* $x^2 = 2y$ *at* P *(first quadrant). Tangent to* C_1 *at* P *touches* C_2, C_3 *(radius* $2\sqrt{3}$*, centres* Q_2, Q_3 *on the y-axis) at* R_2, R_3.

Solving gives $P = (\sqrt{2}, 1)$; tangent line $\sqrt{2}x + y = 3$. Centres satisfy $|c-3|/\sqrt{3} = 2\sqrt{3} \Rightarrow c = 9$ or -3, so $Q_2Q_3 = 12$ (A true). The tangent points (feet of perpendiculars) are $R_2 = (-2\sqrt{2}, 7)$, $R_3 = (2\sqrt{2}, -1)$, giving $R_2R_3 = \sqrt{32+64} = 4\sqrt{6}$ (B true). Area of $\triangle OR_2R_3 = \frac{1}{2}|(-2\sqrt{2})(-1) - (2\sqrt{2})(7)| = 6\sqrt{2}$ (C true). Area of $\triangle PQ_2Q_3 = \frac{1}{2}(12)(\sqrt{2}) = 6\sqrt{2} \neq 4\sqrt{2}$ (D false).
Answer: (A), (B), (C)

Q.13

Unit circle $x^2 + y^2 = 1$, diameter RS with $S = (1, 0)$. P variable point, tangents at S, P meet at Q; normal at P meets the line through Q parallel to RS at E. Locus of E passes through which point(s)?

With $P = (\cos\theta, \sin\theta)$: $Q = (1, \frac{1-\cos\theta}{\sin\theta})$, and working through the intersection gives $E = \left(\dfrac{\cos\theta}{1+\cos\theta}, \dfrac{1-\cos\theta}{\sin\theta}\right)$. Checking each option's x-coordinate against $\cos\theta = x/(1-x)$ and computing the corresponding y: $(1/3, 1/\sqrt{3})$ and $(1/3, -1/\sqrt{3})$ both check out (at $\theta = \pm\pi/3$), while $(1/4, \pm 1/2)$ do not (the actual y value there is $\pm 1/\sqrt{2}$).
Answer: (A), (C)

Q.14

$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$. *Number of distinct real x.*

Row-reducing ($R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$) and expanding along column 1 gives the determinant $= 12x^6 + 2x^3$. Setting $6x^6 + x^3 - 5 = 0$ and solving as a quadratic in x^3 gives $x^3 = 5/6$ or $x^3 = -1$, each with a unique real cube root.
Answer: 2

Q.15

m = smallest positive integer such that the coefficient of x^2 in $(1+x)^2 + \cdots + (1+x)^{49} + (1+mx)^{50}$ is $(3n+1)\binom{51}{3}$ for some positive integer n. Find n.

The coefficient is $\binom{50}{3} + m^2\binom{50}{2} = 19600 + 1225m^2$ (using the hockey-stick identity), and $\binom{51}{3} = 20825 = 1225 \times 17$. Writing $19600 = 1225 \times 16$, the condition reduces to $17 \mid (16 + m^2)$, i.e. $m \equiv \pm 1 \pmod{17}$. The smallest positive m is 1, but this gives $n = 0$ (not positive); the next is $m = 16$, giving $(16 + 256)/17 = 16 = 3n + 1 \Rightarrow n = 5$.
Answer: 5

Q.16

Number of distinct $x \in [0, 1]$ for which $\displaystyle\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$.

Let $F(x) = \int_0^x \frac{t^2}{1+t^4}dt - 2x + 1$. $F(0) = 1 > 0$ and $F(1) \leq \frac{1}{3} - 1 < 0$, so a root exists by the IVT. Since $F'(x) = \frac{x^2}{1+x^4} - 2 \leq \frac{1}{2} - 2 < 0$ always (using $1 + x^4 \geq 2x^2$), F is strictly decreasing, so the root is unique.
Answer: 1

Q.17

$\lim_{x\to 0} \dfrac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. *Find* $6(\alpha + \beta)$.

If $\alpha \neq 1$ the denominator is $O(x)$ while the numerator is $O(x^3)$, giving limit $0 \neq 1$; so $\alpha = 1$, making the denominator $x - \sin x \sim x^3/6$. The numerator $\sim \beta x^3$, so the limit is $6\beta = 1 \Rightarrow \beta = 1/6$.
Answer: 7

Q.18

$z = \dfrac{-1+\sqrt{3}i}{2} = \omega$ *(primitive cube root of unity),* $r, s \in \{1, 2, 3\}$. $P = \begin{pmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{pmatrix}$. *Find the number of* (r, s) *with* $P^2 = -I$.

Writing $P = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $P^2 = -I$ requires $b(a+d) = 0$ and $a^2 + b^2 = -1$. Since $b = \omega^{2s} \neq 0$ always, we need $a = -d$, i.e. $(-1)^r \omega^r = -\omega^r \Rightarrow r$ odd ($r = 1$ or 3). Checking $a^2 + b^2 = -1$ for each: only $r = 1, s = 1$ satisfies $\omega^2 + \omega = -1$ (a known identity); all other combinations fail (e.g. $r = 3$ forces $\omega^{4s} = -2$, impossible since $|\omega^{4s}| = 1$).
Answer: 1