QuestIQ Academy

JEE Advanced 2017 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = paragraph-based, single correct.

Q.1

Plane through $(1,1,1)$ *perpendicular to* $2x+y-2z=5$ *and* $3x-6y-2z=7$.

The required normal is $n_1 \times n_2 = (2,1,-2)\times(3,-6,-2) = (-14,-2,-15)$. The plane through $(1,1,1)$ with this normal is $-14(x-1)-2(y-1)-15(z-1)=0$, i.e. $14x+2y+15z=31$.
Answer: (C) $14x+2y+15z=31$

Q.2

O origin, $\triangle PQR$ *arbitrary, S satisfies* $\vec{OP}\cdot\vec{OQ}+\vec{OR}\cdot\vec{OS} = \vec{OR}\cdot\vec{OP}+\vec{OQ}\cdot\vec{OS} = \vec{OQ}\cdot\vec{OR}+\vec{OP}\cdot\vec{OS}$.

The first equality rearranges to $(\vec{OP}-\vec{OS})\cdot(\vec{OQ}-\vec{OR})=0$, i.e. $\vec{SP}\perp\vec{RQ}$ – so S lies on the altitude from P. The remaining equality similarly gives perpendicularity to another side. A point lying on two altitudes of a triangle is the orthocentre.
Answer: (D) orthocenter

Q.3

$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx$, $x>0$, $y(0)=\sqrt{7}$. *Find* $y(256)$.

Substituting $t=\sqrt{9+\sqrt{x}}$ (so $\sqrt{x}=t^2-9$, $dx=4t(t^2-9)\,dt$) collapses the equation to $dy=\dfrac{dt}{2\sqrt{4+t}}$, integrating to $y=\sqrt{4+t}+C=\sqrt{4+\sqrt{9+\sqrt{x}}}+C$. At $x=0$, $t=3$ gives $y=\sqrt{7}+C=\sqrt{7}\Rightarrow C=0$. At $x=256$: $\sqrt{x}=16$, $9+16=25$, $\sqrt{25}=5$, $4+5=9$, $\sqrt{9}=3$.
Answer: (A) 3

Q.4

$f:\mathbb{R}\to\mathbb{R}$ *twice differentiable,* $f''(x)>0$ *for all* x, $f(\frac{1}{2})=\frac{1}{2}$, $f(1)=1$. *Find the range for* $f'(1)$.

Since $f'' > 0$, f' is strictly increasing. The secant slope between $x = \frac{1}{2}$ and $x = 1$ is $\dfrac{1 - 1/2}{1 - 1/2} = 1$; by the Mean Value Theorem some $c \in (\frac{1}{2}, 1)$ has $f'(c) = 1$. Since f' is strictly increasing and $1 > c$, $f'(1) > f'(c) = 1$.
Answer: (D) $f'(1) > 1$

Q.5

3×3 matrices M with entries from $\{0, 1, 2\}$; how many have $\text{trace}(M^T M) = 5$?

$\text{trace}(M^T M)$ equals the sum of squares of all 9 entries. With squares from $\{0, 1, 4\}$, the only ways to total 5 are: five entries equal to 1 and the rest 0 ($\binom{9}{5} = 126$ placements), or one entry equal to 2 (contributing 4) and one entry equal to 1 (contributing 1), rest 0 ($9 \times 8 = 72$ placements).
Answer: (B) 198

Q.6

$S = \{1, \ldots, 9\}$; N_k = number of 5-element subsets with exactly k odd elements. Find $N_1 + \cdots + N_5$.

S has 5 odd and 4 even elements, so $N_k = \binom{5}{k}\binom{4}{5-k}$, valid for $k \geq 1$ (since only 4 evens exist). This sum equals the total count of 5-element subsets of the 9-element set (the $k = 0$ term is automatically zero, since $\binom{4}{5} = 0$), i.e. $\binom{9}{5}$.
Answer: (D) 126

Q.7

Nonnegative integers x, y, z satisfy $x + y + z = 10$ (chosen uniformly at random among all solutions). Probability z is even?

Total solutions: $\binom{12}{2} = 66$. For each even $z \in \{0, 2, 4, 6, 8, 10\}$, the number of (x, y) pairs is $11 - z$, giving $11 + 9 + 7 + 5 + 3 + 1 = 36$ favourable outcomes.
Answer: (B) 6/11

Q.8

Erratum: Q.44 ($g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1}(t)\, dt$) and Q.45 ($2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$) were both voided by the exam board "due to internal review" – full marks were awarded to all candidates for both.

For reference: by Leibniz's rule, $g'(x) = 2\cos 2x \cdot \sin^{-1}(\sin 2x) - \cos x \cdot \sin^{-1}(\sin x)$; evaluated literally at $x = \pm\pi/2$ this gives 0, not $\pm 2\pi$ – the intended solution path apparently assumed $\sin^{-1}(\sin\theta) = \theta$ without the domain restriction that makes this valid, which is almost certainly why the question was withdrawn. For Q.45, half-angle substitution gives $\tan^2(\alpha/2) = 3\tan^2(\beta/2)$, which factors as $(\tan\frac{\alpha}{2} - \sqrt{3}\tan\frac{\beta}{2})(\tan\frac{\alpha}{2} + \sqrt{3}\tan\frac{\beta}{2}) = 0$ – satisfying option (A) or (C) depending on sign, with no way to determine which from the given data alone.
Answer: Both voided by the exam board – full marks awarded to all candidates

Q.9

$f : \mathbb{R} \to \mathbb{R}$ differentiable, $f'(x) > 2f(x)$ for all x, $f(0) = 1$.

Let $h(x) = e^{-2x}f(x)$; then $h'(x) = e^{-2x}[f'(x) - 2f(x)] > 0$, so h is strictly increasing, with $h(0) = 1$. For $x > 0$: $h(x) > 1 \Rightarrow f(x) > e^{2x}$ (C true). Then $f'(x) > 2f(x) > 2e^{2x} > 0$, so f is increasing on $(0, \infty)$ (A true, B false). Since $f'(x) > 2e^{2x} > e^{2x}$, option (D) is false.

Answer: (A), (C)

Q.10

$f(x) = \dfrac{1 - x(1 + |1 - x|)}{|1 - x|} \cos\left(\dfrac{1}{1 - x}\right)$, $x \neq 1$.

For $x < 1$: $|1 - x| = 1 - x$, and the fraction simplifies to $\dfrac{(1 - x)^2}{1 - x} = 1 - x$, so $f(x) = (1 - x)\cos\left(\frac{1}{1-x}\right) \to 0$ as $x \to 1^-$ (bounded cosine times a factor $\to 0$) – (A) true. For $x > 1$: $|1 - x| = x - 1$, and the fraction simplifies to $-(x + 1)$, so $f(x) = -(x + 1)\cos\left(\frac{1}{1-x}\right)$; as $x \to 1^+$, $\frac{1}{1-x} \to -\infty$ and the cosine oscillates without settling, while $-(x + 1) \to -2 \neq 0$ – the limit does not exist (D true).

Answer: (A), (D)

Q.11

$f(x) = \begin{vmatrix} \cos 2x & \cos 2x & \sin 2x \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$.

Expanding the determinant simplifies remarkably to $f(x) = \cos 2x + \cos 4x$. Then $f'(x) = -2\sin 2x - 4\sin 4x = -2\sin 2x(1 + 4\cos 2x)$, which vanishes when $\sin 2x = 0$ (giving $x = 0, \pm\pi/2$ in $(-\pi, \pi)$: 3 points) or $\cos 2x = -\frac{1}{4}$ (giving 4 more points in $(-\pi, \pi)$) – 7 points total, more than three (B true). Evaluating f at the critical points: $f(0) = 2$ (the largest value, since the $\cos 2x = -1/4$ points give $f = -9/8$ and the other sine-zero points give $f = 0$), so f attains its maximum at $x = 0$ (C true, D false).

Answer: (B), (C)

Q.12

$R = \{(x, y) : x^3 \leq y \leq x,\ 0 \leq x \leq 1\}$; the line $x = \alpha$ splits R's area in half.

Total area $= \int_0^1 (x - x^3)dx = \frac{1}{4}$. Setting $\int_0^\alpha (x - x^3)dx = \frac{1}{8}$ gives $\frac{\alpha^2}{2} - \frac{\alpha^4}{4} = \frac{1}{8}$, i.e. $2\alpha^4 - 4\alpha^2 + 1 = 0$ (C true). Solving for $\alpha^2 = 1 - \frac{1}{\sqrt{2}} \approx 0.293$ gives $\alpha \approx 0.541 \in (\frac{1}{2}, 1)$ (B true).

Answer: (B), (C)

Q.13

$$I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)}\, dx.$$

Since $\frac{k+1}{x+1} \le 1$ for $x \ge k$ (with equality only at $x = k$), $\frac{k+1}{x(x+1)} < \frac{1}{x}$ on $(k, k+1)$, so $I < \sum_{k=1}^{98} \int_k^{k+1} \frac{dx}{x} = \ln 99$ (B true, telescoping). Also $\frac{k+1}{x} \ge 1$ for $x \le k+1$, giving $\frac{k+1}{x(x+1)} \ge \frac{1}{x+1}$, so $I \ge \sum \int_k^{k+1} \frac{dx}{x+1} = \ln 50 \approx 3.9$, which comfortably exceeds $49/50$ (D true, C false).

Answer: (B), (D)

Q.14

$\vec{OX}, \vec{OY}, \vec{OZ}$ are unit vectors along sides $\vec{QR}, \vec{RP}, \vec{PQ}$ of $\triangle PQR$. Find $|\vec{OX} \times \vec{OY}|$.

Traversing the triangle, consecutive directed sides turn through the exterior angle at the shared vertex, so the angle between $\vec{QR}$ and $\vec{RP}$ is $\pi - R$. Thus $|\vec{OX} \times \vec{OY}| = \sin(\pi - R) = \sin R = \sin(\pi - (P+Q)) = \sin(P+Q)$.

Answer: (A) $\sin(P+Q)$

Q.15

Minimum value of $\cos(P+Q) + \cos(Q+R) + \cos(R+P)$ as $\triangle PQR$ varies.

Since $P+Q+R = \pi$, each term equals $-\cos$(third angle), so the sum is $-(\cos P + \cos Q + \cos R)$. The maximum of $\cos P + \cos Q + \cos R$ over all triangles is $\frac{3}{2}$ (attained at the equilateral triangle), so the minimum of the given expression is $-\frac{3}{2}$.

Answer: (B) $-3/2$

Q.16

α, β roots of $x^2 - x - 1 = 0$; $a_n = p\alpha^n + q\beta^n$. Express a_{12}.

Since $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, both α^n and β^n (hence a_n) satisfy the Fibonacci-type recurrence $a_n = a_{n-1} + a_{n-2}$.

Answer: (B) $a_{12} = a_{11} + a_{10}$

Q.17

If $a_4 = 28$, find $p + 2q$.

Using $a_0 = p + q$, $a_1 = \frac{p+q}{2} + \frac{(p-q)\sqrt{5}}{2}$ (from $\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$) and the recurrence $a_n = a_{n-1} + a_{n-2}$, direct computation gives $a_4 = \frac{7(p+q)}{2} + \frac{3(p-q)}{2}\sqrt{5}$. Setting this equal to the rational number 28 and applying the given FACT (a rational plus an irrational multiple of $\sqrt{5}$ is zero only if both parts vanish) forces $\frac{3(p-q)}{2} = 0$ and $\frac{7(p+q)}{2} = 28$, giving $p = q = 4$.

Answer: (D) 12