QuestIQ Academy

JEE Advanced 2018 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = one or more correct; Section 2 = numerical value; Section 3 = list-match (single correct option).

Q.1

$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\dfrac{1}{1+(x+j)(x+j-1)}\right)$.

Each term telescopes: $\tan^{-1}(x+j) - \tan^{-1}(x+j-1) = \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$, so $f_n(x) = \tan^{-1}(x+n) - \tan^{-1} x$, giving $f_n(0) = \tan^{-1} n$. (A) $\sum_{j=1}^{5} \tan^2(\tan^{-1} j) = 1+4+9+16+25 = 55$, true. (B) $f_n'(0) = \frac{1}{1+n^2} - 1$, so $(1+f_j'(0))\sec^2(f_j(0)) = \frac{1}{1+j^2}\cdot(1+j^2) = 1$; summing $j=1$ to 10 gives 10, true. (C) $\tan(f_n(x)) \to 0$ as $x \to \infty$ (not $1/n$), false. (D) $\sec^2(f_n(x)) = 1 + \tan^2(f_n(x)) \to 1$, true.

Answer: (A), (B), (D)

Q.2

T through $P(-2,7), Q(2,-5)$; circles S_1, S_2 tangent to T at P, Q and to each other at M; E_1 = locus of M. F_2 = chords of E_1 through $R(1,1)$; E_2 = their midpoints.

E_1 is the circle on PQ as diameter, excluding P, Q: centre $(0,1)$, radius $2\sqrt{10}$ – so $P = (-2,7)$ itself is excluded from E_1 (A false). Since R is interior to E_1, E_2 is the circle on segment (centre of E_1)–R as diameter, i.e. centre $(0.5, 1)$, radius 0.5, excluding the two midpoints corresponding to chords through the excluded points P, Q. The chord through R and P has midpoint exactly $(4/5, 7/5)$ (computed as the foot of perpendicular from the centre), so this point is excluded from E_2 (B true). The point $(1/2, 1)$ is the *centre* of E_2's circle, not a point on it, so it does not lie in E_2 (C false). $(0, 3/2)$ does not even satisfy E_1's circle equation, so it trivially does not lie in E_1 (D true).

Answer: (B), (D)

Q.3

$S = \{(b_1, b_2, b_3) : -x+2y+5z = b_1,\ 2x-4y+3z = b_2,\ x-2y+2z = b_3$ *solvable*$\}$.

Elimination gives the compatibility condition $b_1 + 7b_2 - 13b_3 = 0$ for S. (A) has coefficient matrix with determinant $12 \neq 0$ – solvable for every (b_1, b_2, b_3), in particular for all of S, true. (B)

reduces to compatibility $b_1+b_2+3b_3=0$, a different plane from S's – not solvable for all of S, false. (C) has rank 1 (all rows proportional), requiring the much more restrictive $b_2=-2b_1$, $b_3=-b_1$ – false. (D) has determinant $54\neq 0$, solvable for every (b_1,b_2,b_3), true.
Answer: (A), (D)

Q.4

Common tangents to $x^2+y^2=\frac{1}{2}$ and $y^2=4x$ meet at Q; ellipse centred at O, semi-major axis OQ, minor axis $\sqrt{2}$.

Tangent to the parabola: $y=mx+1/m$; tangency to the circle gives $m^4+m^2-2=0 \Rightarrow m^2=1$. The two tangents $y=\pm(x+1)$ meet at $Q=(-1,0)$, so $a=OQ=1$; minor semi-axis $b=1/\sqrt{2}$. Eccentricity $e=\sqrt{1-b^2/a^2}=1/\sqrt{2}$, latus rectum $=2b^2/a=1$ (A true, B false). Area between $x=1/\sqrt{2}$ and $x=1$ under $x^2+2y^2=1$: $\sqrt{2}\int_{1/\sqrt{2}}^{1}\sqrt{1-x^2}\,dx=\frac{1}{4\sqrt{2}}(\pi-2)$ (C true, D false).
Answer: (A), (C)

Q.5

$sz+t\bar{z}+r=0$, $s,t,r\neq 0$; L = solution set.

As a real-linear map on $z=x+iy$, $z\mapsto sz+t\bar{z}$ has determinant $|s|^2-|t|^2$. If $|s|\neq|t|$ the map is invertible and L has exactly one point; conversely, whenever $|s|=|t|$ the map has rank exactly 1 (since s,t aren't both zero), so any nonempty solution set is a full line – never a single point. Hence exactly-one-element $\Rightarrow |s|\neq|t|$ (A true). But $|s|=|t|$ can also give an *empty* L (if $-r$ isn't in the range), so (B) is not guaranteed, false. Since L is always empty, a point, or a line, its intersection with any circle has at most 2 points (C true). If L has more than one element it cannot be a single point, so it must be the line case – infinitely many (D true).
Answer: (A), (C), (D)

Q.6

$\lim_{t\to x}\dfrac{f(x)\sin t-f(t)\sin x}{t-x}=\sin^2 x$ *on* $(0,\pi)$*;* $f(\pi/6)=-\pi/12$.

The limit equals $f(x)\cos x-f'(x)\sin x$ (a derivative-in-t evaluated at $t=x$), so $f(x)\cos x-f'(x)\sin x=\sin^2 x$, i.e. $\frac{d}{dx}\left[\frac{f(x)}{\sin x}\right]=-1$. Integrating: $f(x)=\sin x\,(C-x)$; using $f(\pi/6)=-\pi/12$ gives $C=0$, so $f(x)=-x\sin x$. (A) $f(\pi/4)=-\frac{\pi}{4\sqrt{2}}$, not $+\frac{\pi}{4\sqrt{2}}$, false. (B) Since $x-\sin x<x^3/6$ for all $x>0$ (shown via two derivatives), $f(x)=-x\sin x<x^4/6-x^2$ holds throughout $(0,\pi)$, true. (C) $f'(x)=-\sin x-x\cos x$ is negative near 0 and positive at π, so it vanishes somewhere in between, true. (D) $f''(x)=-2\cos x+x\sin x$; $f''(\pi/2)+f(\pi/2)=\frac{\pi}{2}+\left(-\frac{\pi}{2}\right)=0$, true.
Answer: (B), (C), (D)

Q.7

$$\int_0^{1/2}\frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{1/4}}\,dx.$$

The integrand simplifies to $(1+\sqrt{3})(1+x)^{-1/2}(1-x)^{-3/2}$, which is exactly the derivative of $F(x)=\sqrt{\dfrac{1+x}{1-x}}$. So the integral is $(1+\sqrt{3})\big[F(x)\big]_0^{1/2}=(1+\sqrt{3})(\sqrt{3}-1)=3-1$.
Answer: 2.00

Q.8

3×3 *matrix with entries in* $\{-1,0,1\}$*; maximum possible determinant.*

The matrix $\begin{pmatrix}1 & 1 & 1\\ -1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix}$ achieves determinant 4, which is the known maximum for a 3×3 matrix with entries restricted to $\{-1,0,1\}$.
Answer: 4.00

Q.9

$|X|=5, |Y|=7$; $\alpha=$ *one-one functions* $X\to Y$, $\beta=$ *onto functions* $Y\to X$. *Find* $\frac{1}{5!}(\beta-\alpha)$.

$\alpha=P(7,5)=2520$. By inclusion–exclusion, $\beta=\sum_{k=0}^{5}(-1)^k\binom{5}{k}(5-k)^7=16800$. So $\frac{1}{120}(16800-2520)=\frac{14280}{120}$.
Answer: 119.00

Q.10

$dy/dx=(2+5y)(5y-2)$, $f(0)=0$. *Find* $\lim_{x\to-\infty}f(x)$.

Since $(2+5y)(5y-2)=25y^2-4$, separating and integrating (using $f(0)=0$) gives $y=\dfrac{2(1-e^{20x})}{5(1+e^{20x})}$. As $x\to-\infty$, $e^{20x}\to 0$, so $y\to 2/5$.
Answer: 0.40

Q.11

$f(0)=1$, $f(x+y)=f(x)f'(y)+f'(x)f(y)$. *Find* $\ln(f(4))$.

Setting $x=y=0$ gives $f'(0)=1/2$. Differentiating in y and setting $y=0$ gives $f'(x)=2f''(0)f(x)$, a first-order linear relation forcing $f(x)=e^{kx}$ with $k=f'(0)=1/2$. So $f(x)=e^{x/2}$ (verified directly in the original equation), and $f(4)=e^2$.
Answer: 2.00

Q.12

P in the first octant; its image Q in the plane $x+y=3$ lies on the z-axis; distance of P from the x-axis is 5; $R=$ image of P in the xy-plane. Find PR.

Reflection across $x+y=3$ preserves z and requires $a=b$ (for $P=(a,b,c)$) with midpoint on the plane: $a+b=6\Rightarrow a=b=3$. Distance from the x-axis: $\sqrt{b^2+c^2}=5\Rightarrow c=4$. So $P=(3,3,4)$, $R=(3,3,-4)$, and $PR=8$.
Answer: 8.00

Q.13

Unit cube in the first octant; S = centre, T = vertex opposite O; $\vec{p}=\vec{SP}, \vec{q}=\vec{SQ}, \vec{r}=\vec{RS}, \vec{t}=\vec{ST}$. Find $|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})|$.

With $O=(0,0,0)$, $P=(1,0,0)$, $Q=(0,1,0)$, $R=(0,0,1)$, $S=(0.5,0.5,0.5)$, $T=(1,1,1)$: direct computation gives $\vec{p}\times\vec{q}=(0.5,0.5,0)$ and $\vec{r}\times\vec{t}=(0.5,-0.5,0)$, whose cross product is $(0,0,-0.5)$.

Answer: 0.50

Q.14

$X=\sum_{r=1}^{10} r\binom{10}{r}^2$. Find $X/1430$.

Using the identity $\sum_{r=0}^{n} r\binom{n}{r}^2 = n\binom{2n-1}{n-1}$: $X=10\binom{19}{9}=10(92378)=923780$.

Answer: 646.00

Q.15

$E_1=\{x\neq 1 : x/(x-1)>0\}$, $f(x)=\ln(x/(x-1))$; $E_2=\{x\in E_1 : \sin^{-1}(f(x))$ real$\}$, $g=\sin^{-1}\circ f$ on E_2. Match List-I with List-II.

$E_1=(-\infty,0)\cup(1,\infty)$. On $(1,\infty)$, $x/(x-1)$ decreases from ∞ to 1, so f ranges over $(0,\infty)$; on $(-\infty,0)$, $x/(x-1)$ increases from 0 to 1, so f ranges over $(-\infty,0)$. Hence range of $f=(-\infty,0)\cup(0,\infty)$ (P→4). Requiring $f(x)\in[-1,1]$ gives $E_2=\left(-\infty,\frac{1}{1-e}\right]\cup\left[\frac{e}{e-1},\infty\right)$, which is contained in E_1 (R→1) and is exactly the domain of g (S→1). On E_2, f attains every value in $[-1,1]$, so g's range is exactly $[-\pi/2,\pi/2]\approx[-1.57,1.57]$, which contains $(0,1)$ (Q→2).

Answer: P → 4; Q → 2; R → 1; S → 1

Q.16

6 boys, 5 girls; committees α_1 (5 members, exactly 3 boys+2 girls), α_2 (at least 2 members, equal boys/girls), α_3 (5 members, at least 2 girls), α_4 (4 members, at least 2 girls, not both M_1, G_1).

$\alpha_1=\binom{6}{3}\binom{5}{2}=200$. $\alpha_2=\sum_{k=1}^{5}\binom{6}{k}\binom{5}{k}=30+150+200+75+6=461$. $\alpha_3=\binom{11}{5}-\binom{6}{5}-\binom{5}{1}\binom{6}{4}=462-6-75=381$. For α_4: total with ≥ 2 girls (4 members) $=150+60+5=215$; subtract cases with M_1, G_1 both present and ≥ 2 girls overall, which needs ≥ 1 more girl among 2 further picks from the remaining 9 people: $\binom{9}{2}-\binom{5}{2}=36-10=26$. So $\alpha_4=215-26=189$.

Answer: P(α_1) → 200; Q(α_2) → 461; R(α_3) → 381; S(α_4) → 189

Q.17

Hyperbola $H: x^2/a^2-y^2/b^2=1$; conjugate axis LM subtends 60° at vertex N; area of $\triangle LMN=4\sqrt{3}$.

$L=(0,b), M=(0,-b), N=(a,0)$ give an isosceles triangle with apex angle 60° – hence equilateral, with side $2b=\sqrt{a^2+b^2}\Rightarrow a^2=3b^2$. Area $=\sqrt{3}\,b^2=4\sqrt{3}\Rightarrow b=2,\ a=2\sqrt{3}$.

Conjugate axis length $2b = 4$; eccentricity $e = \sqrt{1 + b^2/a^2} = 2/\sqrt{3}$; distance between foci $2ae = 8$; latus rectum $2b^2/a = 4/\sqrt{3}$.
Answer: P $\to$ 4; Q $\to$ $2/\sqrt{3}$; R $\to$ 8; S $\to$ $4/\sqrt{3}$

Q.18

$f_1(x) = \sin\big(\sqrt{1 - e^{-x^2}}\big)$; $f_2(x) = |\sin x|/\tan^{-1} x$ ($f_2(0) = 1$); $f_3(x) = [\sin(\ln(x+2))]$; $f_4(x) = x^2 \sin(1/x)$ ($f_4(0) = 0$). Classify each at $x = 0$.

Near 0, $\sqrt{1 - e^{-x^2}} \approx |x|$, so $f_1(x) \approx \sin|x| \approx |x|$ – continuous but not differentiable at 0 (corner). As $x \to 0^{\pm}$, $|\sin x|/\tan^{-1} x \to \pm 1$, both disagreeing with $f_2(0) = 1$ on the left – not continuous. Since $\sin(\ln 2) \approx 0.638$ is not an integer, f_3 is locally constant (equal to 0) near $x = 0$ – differentiable with continuous (zero) derivative. f_4 is the classic example: differentiable at 0 (derivative 0) but $f_4'(x) = 2x\sin(1/x) - \cos(1/x)$ has no limit at 0 – derivative not continuous.
Answer: f_1 $\to$ continuous, not differentiable; f_2 $\to$ not continuous; f_3 $\to$ differentiable, derivative continuous; f_4 $\to$ differentiable, derivative not continuous