QuestIQ Academy

JEE Advanced 2019 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = numerical value.

Q.1

$S = \{z : |z - 2 + i| \geq \sqrt{5}\}$. *If* z_0 *maximizes* $\frac{1}{|z_0 - 1|}$ *over* S, *find* $\arg\left(\dfrac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}\right)$.

Maximizing $1/|z-1|$ means minimizing $|z-1|$ over S, the exterior of the circle centred $(2, -1)$, radius $\sqrt{5}$. Since $(1, 0)$ is inside this circle (distance $\sqrt{2} < \sqrt{5}$), the closest boundary point lies along the ray from the centre through $(1, 0)$, extended to the circle: $z_0 = (2 - \sqrt{5/2}) + i(\sqrt{5/2} - 1)$. With $t = \sqrt{5/2}$: $4 - z_0 - \bar{z}_0 = 2t$ and $z_0 - \bar{z}_0 + 2i = 2it$, so the ratio is $1/i = -i$, with argument $-\pi/2$.

Answer: (A) $-\pi/2$

Q.2

$M = \begin{pmatrix} \sin^4\theta & -1 - \sin^2\theta \\ 1 + \cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha I + \beta M^{-1}$. *Find* $\alpha^* + \beta^*$ *(min values).*

By Cayley–Hamilton, $M^{-1} = (\mathrm{tr}M \cdot I - M)/\det M$, forcing $\alpha = \mathrm{tr}M$, $\beta = -\det M$. With $s = \sin^2\theta\cos^2\theta \in [0, \frac{1}{4}]$: $\alpha = 1 - 2s$ (min $\frac{1}{2}$ at $s = \frac{1}{4}$) and $\det M = s^2 + s + 2$ so $\beta = -(s^2 + s + 2)$ (min $-\frac{37}{16}$ at $s = \frac{1}{4}$). Sum $= \frac{1}{2} - \frac{37}{16} = -\frac{29}{16}$.

Answer: (C) $-29/16$

Q.3

Line $y = mx + 1$ *meets circle* $(x-3)^2 + (y+2)^2 = 25$ *at* P, Q; *midpoint of* PQ *has x-coordinate* $-3/5$. *Find the range containing* m.

The centre-to-midpoint segment is perpendicular to the chord. With midpoint (h, k), $h = -3/5$, using $k = -\frac{3m}{5} + 1$ and the perpendicularity condition gives $m^2 - 5m + 6 = 0$, so $m = 2$ or $m = 3$, both in $[2, 4)$.

Answer: (B) $2 \leq m < 4$

Q.4

Area of $\{(x,y): xy \le 8,\ 1 \le y \le x^2\}$.

Curves $y = x^2$ and $y = 8/x$ meet at $x = 2$. For $1 \le x \le 2$ the upper bound is x^2; for $2 \le x \le 8$ it is $8/x$.

$$\int_1^2 (x^2 - 1)\,dx + \int_2^8 \left(\frac{8}{x} - 1\right) dx = \frac{4}{3} + (16\ln 2 - 6) = 16\ln 2 - \frac{14}{3}$$

Answer: (A) $16\log_e 2 - 14/3$

Q.5

$a_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, *$\alpha, \beta$ roots of $x^2 - x - 1 = 0$; $b_1 = 1$, $b_n = a_{n-1} + a_{n+1}$ $(n \ge 2)$.*

Here a_n is the Fibonacci sequence $(1, 1, 2, 3, 5, \dots)$ and $b_n = \alpha^n + \beta^n$ is the Lucas sequence. (A) The Fibonacci partial-sum identity $\sum_{i=1}^n a_i = a_{n+2} - 1$ holds for all n. (B) The generating function $\sum a_n x^n = \frac{x}{1-x-x^2}$ at $x = 1/10$ gives $\frac{10}{89}$. (C) is the definition of b_n. (D) $\sum b_n/10^n = 12/89 \neq 8/89$.

Answer: (A), (B), (C)

Q.6

$M = \begin{pmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{pmatrix}$, $\operatorname{adj} M = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$.

Matching cofactors independent of a, b confirms consistency; solving gives $a = 2, b = 1$, so (A) $a + b = 3$ is true. Then $\det M = -2$. Since $(\operatorname{adj} M)^{-1} = M/\det M$ and $\operatorname{adj}(M^{-1}) = M/\det M$ also, their sum is $2M/\det M = -M$, so (B) is true. $\det(\operatorname{adj}(M^2)) = (\det M)^4 = 16 \neq 81$, so (C) is false. Solving $Mx = (1, 2, 3)^T$ gives $\alpha = 1, \beta = -1, \gamma = 1$, so $\alpha - \beta + \gamma = 3$, (D) true.

Answer: (A), (B), (D)

Q.7

Bags B_1, B_2, B_3 chosen with probabilities $0.3, 0.3, 0.4$; B_1: 5R5G, B_2: 3R5G, B_3: 5R3G. A ball is drawn.

(A) $P(G \mid B_3) = 3/8$, true. (B) $P(G) = 0.3(0.5) + 0.3(5/8) + 0.4(3/8) = 39/80$, true. (C) $P(B_3 \mid G) = \frac{0.4 \times 3/8}{39/80} = 4/13 \neq 5/13$, false. (D) $P(B_3 \cap G) = 0.4 \times 3/8 = 3/20 \neq 3/10$, false.

Answer: (A), (B)

Q.8

$\triangle PQR$: $p = \sqrt{3}$, $q = 1$, circumradius $= 1$, not right-angled. Median from R meets PQ at S; perpendicular from P meets QR at E; RS, PE meet at O.

Law of sines forces $P = 120°, Q = 30°, R = 30°$ (the $90°$ case is excluded), giving isosceles $PQ = PR = 1$, $QR = \sqrt{3}$. Placing $Q = (-\frac{\sqrt{3}}{2}, 0)$, $R = (\frac{\sqrt{3}}{2}, 0)$, $P = (0, \frac{1}{2})$: $S = (-\frac{\sqrt{3}}{4}, \frac{1}{4})$, $E = (0, 0)$, and line RS meets $x = 0$ at $O = (0, \frac{1}{6})$. Then $RS = \sqrt{7}/2$ (A, true); $OE = 1/6$ (C,

true); the incircle radius $= \frac{\text{Area}}{s} = \frac{2\sqrt{3}-3}{2} = \frac{\sqrt{3}}{2}(2-\sqrt{3})$ (D, true). Direct computation of area $\triangle SOE = \sqrt{3}/48 \neq \sqrt{3}/12$ (B, false).
Answer: (A), (C), (D)

Q.9

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$. R_n *= largest-area rectangle inscribed in* E_n*;* E_{n+1} *= largest-area ellipse inscribed in* R_n*.*

The largest rectangle in an ellipse of semi-axes a, b has half-widths $a/\sqrt{2}, b/\sqrt{2}$ and area $2ab$; the largest ellipse in that rectangle exactly fits it. So $a_n = 3/(\sqrt{2})^{n-1}$, $b_n = 2/(\sqrt{2})^{n-1}$, giving constant eccentricity for all n (A false). Area$(R_n) = 12/2^{n-1}$, a geometric series summing to 24 but never reaching it for finite N (B true). For E_9: $a_9 = 3/16, b_9 = 1/8$; latus rectum $= 2b^2/a = 1/6$ (C true); focal distance $c = \sqrt{5}/16 \neq \sqrt{5}/32$ (D false).
Answer: (B), (C)

Q.10

Piecewise f*: quintic for* $x < 0$*, quadratic on* $[0, 1)$*, cubic on* $[1, 3)$*, log-type for* $x \geq 3$*.*

For $x < 0$, $f(x) = (x+1)^5 - 2x$ so $f'(x) = 5(x+1)^4 - 2$, which is negative on part of $(-\infty, 0)$ – f is not monotonic there (A false). At $x = 1$ the one-sided derivatives of f' agree (both $= 1$) but f'' jumps from 2 to -4, so $x = 1$ is a local maximum of f' (B true) and f' is not differentiable there (D true). Since f is continuous and $f \to -\infty$ as $x \to -\infty$, $f \to +\infty$ as $x \to +\infty$, f is onto by the Intermediate Value Theorem (C true).
Answer: (B), (C), (D)

Q.11

Curve Γ *through* $(1, 0)$ *in the first quadrant; the tangent at any point* P *meets the* y*-axis at* Y_P *with* $PY_P = 1$*.*

If $y = y(x)$, the y-intercept of the tangent is $y - xy'$, so $PY_P^2 = x^2(1 + y'^2) = 1$. The physically valid (first-quadrant, decreasing) branch is $y' = -\sqrt{1-x^2}/x$, whose solution satisfying $y(1) = 0$ is $y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$, which indeed satisfies $xy' + \sqrt{1-x^2} = 0$.
Answer: (A), (B)

Q.12

$L_1 : \vec{r} = \hat{\imath} + \lambda(-\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$, $L_2 : \vec{r} = \mu(2\hat{\imath} - \hat{\jmath} + 2\hat{k})$. L_3 *is the common perpendicular.*

Direction of $L_3 \propto d_1 \times d_2 = (6, 6, -3) \parallel (2, 2, -1)$. Solving for the feet of the common perpendicular gives points $(\frac{8}{9}, \frac{2}{9}, \frac{2}{9})$ on L_1 and $(\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$ on L_2, both lying on the same line of direction $(2, 2, -1)$; the point $(\frac{2}{3}, 0, \frac{1}{3})$ also lies on this line (checked via the parallel-displacement test), but the origin does not.
Answer: (A), (B), (C)

Q.13

Minimum value of $|a+b\omega+c\omega^2|^2$ *over distinct non-zero integers* a,b,c *(*$\omega \neq 1$ *a cube root of unity).*

$|a+b\omega+c\omega^2|^2 = \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$, minimized by any three consecutive non-zero integers (e.g. $1,2,3$), giving differences $1,1,2$ and value $\frac{1}{2}(1+1+4)=3$.

Answer: 3.00

Q.14

$AP(1;3)\cap AP(2;5)\cap AP(3;7) = AP(a;d)$. *Find* $a+d$.

By CRT, $x\equiv 1\ (3),\ x\equiv 2\ (5),\ x\equiv 3\ (7)\ \Rightarrow x\equiv 52 \pmod{105}$. So $a=52,\ d=105$.

Answer: 157.00

Q.15

S *= all* 3×3 $0/1$ *matrices.* E_1*:* $\det = 0$*;* E_2*: entry-sum* $=7$*. Find* $P(E_1 \mid E_2)$*.*

$|E_2| = \binom{9}{3} = 36$ (choosing the 2 zero positions). If the two zeros share a row or a column, the matrix has two equal rows/columns, forcing $\det = 0$ ($9+9=18$ such placements). If the zeros are in different rows and columns (the remaining 18 placements), direct computation gives $\det = \pm 1 \neq 0$ in every case. So $|E_1\cap E_2| = 18$, and $P(E_1 \mid E_2) = 18/36$.

Answer: 0.50

Q.16

$A=(2,3)$ *reflected in* $8x-6y-23=0$ *to give* B*. Circles centred* A,B *with radii* $2,1$*; common tangent* T *meets* AB *at* C*. Find* AC*.*

Distance from A to the mirror line is 2.5, so $AB=5$. C is the external centre of similitude, dividing AB externally in ratio $2:1$, giving $AC = 2\cdot AB = 10$.

Answer: 10.00

Q.17

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}.\ \textit{Find } 27I^2.$$

Using $x\to -x$ symmetry (since $2-\cos 2x$ is even and $\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{-\sin x}}=1$), $I = \frac{2}{\pi}\int_0^{\pi/4}\frac{dx}{2-\cos 2x} = \frac{2}{\pi}\int_0^{\pi/4}\frac{dx}{1+2\sin^2 x}$. Substituting $u=\tan x$ gives $\int_0^1 \frac{du}{1+3u^2} = \frac{\pi}{3\sqrt{3}}$, so $I = \frac{2}{3\sqrt{3}}$ and $27I^2 = 27\cdot\frac{4}{27} = 4$.

Answer: 4.00

Q.18

Lines through the origin with directions $(1,0,0)$, $(1,1,0)$, $(1,1,1)$ *meet the plane* $x+y+z=1$ *at* A,B,C*. Find* $(6\Delta)^2$ *where* Δ *= area of* $\triangle ABC$*.*

$A = (1, 0, 0)$, $B = (\frac{1}{2}, \frac{1}{2}, 0)$, $C = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. $\vec{AB} \times \vec{AC} = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, so $\Delta = \frac{1}{2} \cdot \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{12}$ and $(6\Delta)^2 = (\frac{\sqrt{3}}{2})^2 = \frac{3}{4}$.

Answer: 0.75