QuestIQ Academy

JEE Advanced 2020 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct (4 options); Section 2 = one or more correct; Section 3 = numerical value.

Q.1

Suppose a, b denote the distinct real roots of $x^2 + 20x - 2020$ and c, d denote the distinct complex roots of $x^2 - 20x + 2020$. Find $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$.

Expanding and grouping:

$$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d) = (a^2 + b^2)(c + d) - (c^2 + d^2)(a + b)$$

From the first quadratic: $a + b = -20,\ ab = -2020 \Rightarrow a^2 + b^2 = (a + b)^2 - 2ab = 400 + 4040 = 4440$. From the second: $c + d = 20,\ cd = 2020 \Rightarrow c^2 + d^2 = (c + d)^2 - 2cd = 400 - 4040 = -3640$.

$$= 4440(20) - (-3640)(-20) = 88800 - 72800 = 16000$$

Answer: (D) 16000

Q.2

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|(x - \sin x)$. Which statement is TRUE?

Let $g(x) = x - \sin x$; $g(0) = 0$, $g'(x) = 1 - \cos x \geq 0$ so g is non-decreasing and strictly increasing except at isolated points, hence $g(x) > 0$ for $x > 0$ and $g(x) < 0$ for $x < 0$. Thus $f(x) = x\,g(x) > 0$ for $x > 0$, $f(x) = -x\,g(x) < 0$ for $x < 0$; f is odd. One can show f is strictly increasing on all of $\mathbb{R}$ (its derivative is non-negative and vanishes only at isolated points) and $f(x) \to \pm\infty$ as $x \to \pm\infty$, so f is a bijection.

Answer: (C) f is BOTH one-one and onto

Q.3

$f(x) = e^{x-1} - e^{-|x-1|}$, $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$. Area in the first quadrant bounded by $y = f(x)$, $y = g(x)$, $x = 0$.

Let $t = x - 1$. For $x \leq 1$ ($t \leq 0$): $f(x) = e^t - e^t = 0$. For $x \geq 1$: $f(x) = e^t - e^{-t} = 2 \sinh t$. Also $g(x) = \cosh t$ for all x.

Curves meet where $2\sinh t = \cosh t \Rightarrow e^{2t} = 3 \Rightarrow t_0 = \frac{1}{2}\ln 3$, so $e^{t_0} = \sqrt{3}$, $\sinh t_0 = \frac{1}{\sqrt{3}}$, $\cosh t_0 = \frac{2}{\sqrt{3}}$.

$$\text{Area} = \int_0^1 \cosh(x-1)\,dx + \int_0^{t_0} (\cosh t - 2\sinh t)\,dt$$

First integral $= \sinh(1) - \sinh(-1)\cdot(-1)$... evaluating: $\int_{-1}^0 \cosh u\,du = \sinh(0) - \sinh(-1) = \sinh 1$.

Second integral $= [\sinh t - 2\cosh t]_0^{t_0} = \left(\frac{1}{\sqrt{3}} - \frac{4}{\sqrt{3}}\right) - (0-2) = -\sqrt{3} + 2$.

Total $= \sinh(1) + 2 - \sqrt{3} = (2-\sqrt{3}) + \frac{1}{2}(e - e^{-1})$.

Answer: (A) $(2-\sqrt{3}) + \frac{1}{2}(e - e^{-1})$

Q.4

P is the end of the latus rectum of $y^2 = 4\lambda x$; ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through P; tangents to parabola and ellipse at P are perpendicular. Find eccentricity of ellipse.

$P = (\lambda, 2\lambda)$. Parabola tangent slope at P: $y' = 2\lambda/y = 1$. Ellipse tangent slope at P: $-\dfrac{b^2 x_1}{a^2 y_1} = -\dfrac{b^2}{2a^2}$. Perpendicular: $1 \cdot \left(-\dfrac{b^2}{2a^2}\right) = -1 \Rightarrow b^2 = 2a^2$. Since $b > a$ here, b is the semi-major axis: $e^2 = 1 - \dfrac{a^2}{b^2} = 1 - \dfrac{1}{2} = \dfrac{1}{2}$.

Answer: (A) $\dfrac{1}{\sqrt{2}}$

Q.5

C_1, C_2 biased coins, $P(\text{head}) = \frac{2}{3}, \frac{1}{3}$ resp., each tossed twice. α = heads from C_1, β = heads from C_2. Probability that $x^2 - \alpha x + \beta$ has real equal roots.

Need $\alpha^2 = 4\beta$. With $\alpha, \beta \in \{0,1,2\}$: only $(\alpha,\beta) = (0,0)$ and $(2,1)$ work. $P(\alpha{=}0) = \frac{1}{9}$, $P(\beta{=}0) = \frac{4}{9} \Rightarrow P = \frac{4}{81}$. $P(\alpha{=}2) = \frac{4}{9}$, $P(\beta{=}1) = \frac{4}{9} \Rightarrow P = \frac{16}{81}$. Total $= \frac{4}{81} + \frac{16}{81} = \frac{20}{81}$.

Answer: (B) $\dfrac{20}{81}$

Q.6

Rectangles in $\{0 \le x \le \pi/2,\ 0 \le y \le 2\sin 2x\}$ with one side on the x-axis. Area of the rectangle of maximum perimeter.

By symmetry about $x = \pi/4$, take width $2s$ centred at $\pi/4$; height $= 2\cos(2s)$ (value of curve at the lower endpoint). Perimeter $P(s) = 4s + 4\cos(2s)$. $P'(s) = 4 - 8\sin(2s) = 0 \Rightarrow \sin 2s = \frac{1}{2} \Rightarrow s = \pi/12$ (max, since $P'' < 0$ there). Width $= \pi/6$, height $= 2\cos(\pi/6) = \sqrt{3}$. Area $= \dfrac{\pi}{6}\cdot\sqrt{3} = \dfrac{\pi}{2\sqrt{3}}$.

Answer: (C) $\dfrac{\pi}{2\sqrt{3}}$

Q.7

$f(x) = x^3 - x^2 + (x-1)\sin x$, g arbitrary, $(fg)(x) = f(x)g(x)$. Which statements are TRUE?

$f(x) = (x-1)(x^2 + \sin x)$; let $h(x) = x^2 + \sin x$, $h(1) = 1 + \sin 1 \neq 0$. Since $(fg)(1) = 0$,

$$(fg)'(1) = \lim_{x\to 1} \frac{(x-1)h(x)g(x)}{x-1} = \lim_{x\to 1} h(x)g(x).$$

(A) g continuous at 1 $\Rightarrow h(x)g(x) \to h(1)g(1)$, limit exists $\Rightarrow$ TRUE.
(B) fg differentiable only requires the limit of $h(x)g(x)$ to exist as $x \to 1$; this does not force $g(1)$ to equal that limit, so g need not be continuous at 1 — FALSE.
(C) g differentiable at 1 $\Rightarrow g$ continuous at 1 $\Rightarrow$ (as in A) TRUE.
(D) Existence of $\lim h(x)g(x)$ only forces $\lim g(x)$ to exist, not that g is differentiable — FALSE.
Answer: (A), (C)

Q.8

M is 3×3 invertible, I the identity. If $M^{-1} = adj(adj\ M)$, which statements are ALWAYS TRUE?

For $n \times n$: $\operatorname{adj}(\operatorname{adj} M) = (\det M)^{n-2}M$; for $n = 3$: $= (\det M)M$. So $M^{-1} = (\det M)M \Rightarrow I = (\det M)M^2$. Taking determinants of $\operatorname{adj}(\operatorname{adj} M) = M^{-1}$: $(\det M)^{(n-1)^2} = (\det M)^{-1} \Rightarrow (\det M)^4 = (\det M)^{-1} \Rightarrow (\det M)^5 = 1 \Rightarrow \det M = 1$ (real). Then $I = M^2$, i.e. $M^2 = I$ (this does not force $M = I$; e.g. $M = \operatorname{diag}(1, -1, -1)$ also has det $= 1$, $M^2 = I$, so (A) is false). $\operatorname{adj} M = (\det M)M^{-1} = M^{-1}$, so $(\operatorname{adj} M)^2 = (M^{-1})^2 = (M^2)^{-1} = I$.
Answer: (B), (C), (D)

Q.9

$S = \{z \in \mathbb{C} : |z^2 + z + 1| = 1\}$. *Which statements are TRUE?*

Let $w = z + \frac{1}{2}$, so $z^2 + z + 1 = w^2 + \frac{3}{4}$ and $|w^2 + \frac{3}{4}| = 1$. With $r = |w|$: Triangle inequality: $1 \le r^2 + \frac{3}{4} \Rightarrow r^2 \ge \frac{1}{4} \Rightarrow r \ge \frac{1}{2}$ (so (C) TRUE, (A) FALSE since equality is not an upper bound). Reverse triangle inequality: $|r^2 - \frac{3}{4}| \le 1 \Rightarrow r^2 \le \frac{7}{4} \Rightarrow r \le \frac{\sqrt{7}}{2}$. Then $|z| = |w - \frac{1}{2}| \le r + \frac{1}{2} \le \frac{\sqrt{7}}{2} + \frac{1}{2} \approx 1.82 < 2$, so $|z| \le 2$ always — (B) TRUE. S is a one-parameter curve (one real equation in two real unknowns), so it is infinite, not 4 points — (D) FALSE.
Answer: (B), (C)

Q.10

Triangle with sides x, y, z opposite X, Y, Z. If $\tan\frac{X}{2} + \tan\frac{Z}{2} = \dfrac{2y}{x+y+z}$, which statements are TRUE?

Using $\tan\frac{X}{2} = \dfrac{r}{s-x}$, $\tan\frac{Z}{2} = \dfrac{r}{s-z}$ (with $s = \frac{x+y+z}{2}$):

$$\frac{r\,y}{(s-x)(s-z)} = \frac{y}{s} \;\Rightarrow\; rs = (s-x)(s-z) \;\Rightarrow\; K = (s-x)(s-z)$$

where K is the area. Using Heron's formula $K^2 = s(s-x)(s-y)(s-z)$ and substituting $K = (s-x)(s-z)$ gives $(s-x)(s-z) = s(s-y) = K$, so $K = rs = s(s-y) \Rightarrow r = s - y$. Since $r = (s-y)\tan\frac{Y}{2}$ in general, $\tan\frac{Y}{2} = 1 \Rightarrow Y = \pi/2$.

So the triangle is right-angled at Y (side y is the hypotenuse). Then $X+Z=\pi-Y=\pi/2=Y$, i.e. $Y=X+Z$ — TRUE. With right angle at Y: $\tan X = x/z$ (legs x, z), so $\tan\frac{X}{2}=\dfrac{\sin X}{1+\cos X}=\dfrac{x/y}{1+z/y}=\dfrac{x}{y+z}$ — TRUE. ($2Y=X+Z$ would need $Y=\pi/3$, contradiction; $x^2+z^2-y^2=0\neq xz$.)
Answer: (B), (C)

Q.11

$L_1:\frac{x-1}{1}=\frac{y}{-1}=\frac{z-1}{3}$, $L_2:\frac{x-1}{-3}=\frac{y}{-1}=\frac{z-1}{1}$. $L:\frac{x-\alpha}{l}=\frac{y-1}{m}=\frac{z-\gamma}{-2}$ lies in the plane of L_1, L_2, passes through their intersection, and bisects the acute angle between them.

Both lines pass through $(1,0,1)$. Directions $d_1=(1,-1,3),\ d_2=(-3,-1,1)$, $|d_1|=|d_2|=\sqrt{11}$, $\cos\theta=\dfrac{d_1\cdot d_2}{11}=\dfrac{1}{11}>0$ so the angle between them (as given) is already acute, and its bisector direction is $d_1+d_2=(-2,-2,4)\parallel(1,1,-2)$.

Matching the z-coefficient -2 gives $l=1,\ m=1$. Line L must also pass through $(1,0,1)$:

$$\frac{0-1}{1}=-1\Rightarrow\frac{1-\alpha}{1}=-1\Rightarrow\alpha=2;\qquad\frac{1-\gamma}{-2}=-1\Rightarrow\gamma=-1$$

So $\alpha-\gamma=3$ and $l+m=2$.
Answer: (A), (B)

Q.12

Which inequalities are TRUE? (A) $\int_0^1 x\cos x\,dx\geq\frac{3}{8}$ (B) $\int_0^1 x\sin x\,dx\geq\frac{3}{10}$ (C) $\int_0^1 x^2\cos x\,dx\geq\frac{1}{2}$ (D) $\int_0^1 x^2\sin x\,dx\geq\frac{2}{9}$

$\int_0^1 x\cos x\,dx=[\,x\sin x+\cos x\,]_0^1=\sin 1+\cos 1-1\approx 0.3818\geq 0.375$ — TRUE. $\int_0^1 x\sin x\,dx=[-x\cos x+\sin x]_0^1=\sin 1-\cos 1\approx 0.3012\geq 0.3$ — TRUE. $\int_0^1 x^2\cos x\,dx=[x^2\sin x+2x\cos x-2\sin x]_0^1=2\cos 1-\sin 1\approx 0.2391<0.5$ — FALSE. $\int_0^1 x^2\sin x\,dx=[-x^2\cos x+2x\sin x+2\cos x]_0^1=\cos 1+2\sin 1-2\approx 0.2232\geq 0.2222$ — TRUE.
Answer: (A), (B), (D)

Q.13

m = min of $\log_3(3^{y_1}+3^{y_2}+3^{y_3})$ subject to $y_1+y_2+y_3=9$; M = max of $\log_3 x_1+\log_3 x_2+\log_3 x_3$ subject to $x_1+x_2+x_3=9,\ x_i>0$. Find $\log_2(m^3)+\log_3(M^2)$.

By AM-GM on the positive terms 3^{y_i}: $3^{y_1}+3^{y_2}+3^{y_3}\geq 3\cdot 3^{(y_1+y_2+y_3)/3}=3\cdot 27=81$, equality at $y_i=3$. So $m=\log_3 81=4$. By AM-GM: $x_1x_2x_3\leq\left(\frac{x_1+x_2+x_3}{3}\right)^3=27$, equality at $x_i=3$. So $M=\log_3(x_1x_2x_3)_{\max}=\log_3 27=3$. $\log_2(4^3)+\log_3(3^2)=\log_2 64+\log_3 9=6+2=8$.
Answer: 8

Q.14

$a_1,a_2,\ldots$ AP of positive integers, common difference 2; $b_1,b_2,\ldots$ GP of positive integers, common ratio 2; $a_1=b_1=c$. Number of c for which $2(a_1+\cdots+a_n)=b_1+\cdots+b_n$ for some positive integer n.

Sum of AP terms: $n(c+n-1)$; doubled: $2n(c+n-1)$. Sum of GP terms: $c(2^n-1)$. Equating:

$$c = \frac{2n(n-1)}{2^n - 2n - 1}$$

Testing $n = 1, 2, \ldots$: $n = 1 \Rightarrow c = 0$ (invalid, need positive integer); $n = 2 \Rightarrow c = -4$ (invalid); $n = 3 \Rightarrow c = \dfrac{12}{1} = 12$ (valid!); $n = 4 \Rightarrow c = \frac{24}{7}$; $n = 5 \Rightarrow \frac{40}{21}$; $n = 6 \Rightarrow \frac{60}{51}$ — none integers, and for $n \geq 7$ the value is already below 1 and shrinking. Only $c = 12$ works.
Answer: 1

Q.15

$f(x) = (3 - \sin 2\pi x)\sin(\pi x - \frac{\pi}{4}) - \sin(3\pi x + \frac{\pi}{4})$ on $[0, 2]$. If $\{x : f(x) \geq 0\} = [\alpha, \beta]$, find $\beta - \alpha$.

Let $A = \pi x - \frac{\pi}{4}$. Then $3\pi x + \frac{\pi}{4} = 3A + \pi$, so $\sin(3\pi x + \frac{\pi}{4}) = -\sin 3A$; and $2\pi x = 2A + \frac{\pi}{2}$, so $\sin 2\pi x = \cos 2A$.

$$f = (3 - \cos 2A)\sin A + \sin 3A = (2 + 2\sin^2 A)\sin A + 3\sin A - 4\sin^3 A = 5\sin A - 2\sin^3 A = \sin A(5 - 2\sin^2 A)$$

Since $5 - 2\sin^2 A \geq 3 > 0$ always, $\text{sign}(f) = \text{sign}(\sin A)$. As x runs over $[0, 2]$, A runs over $[-\pi/4, 7\pi/4]$ (length 2π), and $\sin A \geq 0$ exactly for $A \in [0, \pi]$, i.e. $x \in [\frac{1}{4}, \frac{5}{4}]$.
Answer: $\beta - \alpha = \frac{5}{4} - \frac{1}{4} = 1$

Q.16

Triangle PQR: $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$, $\vec{c} = \overrightarrow{PQ}$, $|\vec{a}| = 3, |\vec{b}| = 4$, and $\dfrac{\vec{a}\cdot(\vec{c} - \vec{b})}{\vec{c}\cdot(\vec{a} - \vec{b})} = \dfrac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$. Find $|\vec{a} \times \vec{b}|^2$.

Since $\vec{a} + \vec{b} + \vec{c} = 0$, $\vec{c} = -(\vec{a} + \vec{b})$. $\vec{a}\cdot(\vec{c} - \vec{b}) = \vec{a}\cdot(-\vec{a} - 2\vec{b}) = -9 - 2(\vec{a}\cdot\vec{b})$. $\vec{c}\cdot(\vec{a} - \vec{b}) = -(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = -(|\vec{a}|^2 - |\vec{b}|^2) = 7$.

$$\frac{-9 - 2(\vec{a}\cdot\vec{b})}{7} = \frac{3}{7} \;\Rightarrow\; \vec{a}\cdot\vec{b} = -6$$

$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = 9 \cdot 16 - 36 = 108$.
Answer: 108

Q.17

$S = \{(x^2 - 1)^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_i \in \mathbb{R}\}$. For $f \in S$, let m_g = number of distinct real roots of g. Find the minimum possible value of $m_{f'} + m_{f''}$.

Try $p(x) = x^2 + 1 > 0$ (no real roots): $f(x) = (x^2 - 1)^2(x^2 + 1) = x^6 - x^4 - x^2 + 1$. $f'(x) = 6x^5 - 4x^3 - 2x = 2x(3x^2 + 1)(x^2 - 1)$, distinct real roots $\{-1, 0, 1\}$, so $m_{f'} = 3$. $f''(x) = 30x^4 - 12x^2 - 2$; solving $15u^2 - 6u - 1 = 0$ ($u = x^2$) gives one positive root $u = \frac{3+2\sqrt{6}}{15}$ and one negative (rejected), so $m_{f''} = 2$.

Why this is optimal: since $(x-1)^2(x+1)^2$ is always ≥ 0 and forces $x = \pm 1$ to be roots of f' with $f(\pm 1) = 0$ as local minima whenever $p > 0$ everywhere, continuity forces an interior local

maximum strictly between -1 and 1 — a third root of f' — so $m_{f'} \geq 3$ is unavoidable. With f' having (at least) 3 real roots, Rolle's theorem forces f'' to have at least 2 real roots between consecutive roots of f', so $m_{f''} \geq 2$. Both bounds are attained simultaneously above.
Answer: 5

Q.18

Find a such that $\lim_{x\to 0^+} \dfrac{(1-x)^{1/x} - e^{-1}}{x^a}$ *is a nonzero real number.*

$\ln(1-x) = -x - \frac{x^2}{2} - \cdots \Rightarrow \frac{1}{x}\ln(1-x) = -1 - \frac{x}{2} - \frac{x^2}{3} - \cdots$

$$(1-x)^{1/x} = e^{-1}\exp\left(-\tfrac{x}{2} - \tfrac{x^2}{3} - \cdots\right) = e^{-1}\left(1 - \tfrac{x}{2} + O(x^2)\right)$$

So $(1-x)^{1/x} - e^{-1} \sim -\dfrac{e^{-1}}{2}x$ as $x \to 0^+$. The ratio $/x^a$ tends to a nonzero finite limit only when $a = 1$ (giving limit $-e^{-1}/2$); for $a < 1$ the limit is 0, for $a > 1$ it diverges.
Answer: 1