QuestIQ Academy

JEE Advanced 2021 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct (Q1–4); Section 2 = numerical value (Q5–10); Section 3 = one or more correct (Q11–16); Section 4 = non-negative integer (Q17–19).

Q.1

Triangle with two sides on the x-axis and on $x + y + 1 = 0$; orthocentre $(1, 1)$. Find the circle through the vertices.

The two given sides meet at $A = (-1, 0)$. The altitude from C (perpendicular to the x-axis) is $x = 1$; intersecting with $x + y + 1 = 0$ gives $C = (1, -2)$. The altitude from B (perpendicular to $x + y + 1 = 0$, slope 1) through $(1, 1)$ is $y = x$; intersecting $y = 0$ gives $B = (0, 0)$. Circle through $A(-1, 0), B(0, 0), C(1, -2)$: $x^2 + y^2 + Dx + Ey + F = 0$ with $F = 0, D = 1, E = 3$.
Answer: (B) $x^2 + y^2 + x + 3y = 0$

Q.2

Area of $\{0 \le x \le \frac{9}{4},\ 0 \le y \le 1,\ x \ge 3y,\ x + y \ge 2\}$.

For each x, y ranges over $[\max(0, 2 - x), \min(1, x/3)]$; since $x/3 < 1$ throughout, the upper bound is always $x/3$. The region is non-empty for $x \in [\frac{3}{2}, \frac{9}{4}]$. Splitting at $x = 2$ and integrating gives area $= \frac{1}{6} + \frac{17}{96} = \frac{11}{32}$.
Answer: (A) 11/32

Q.3

$E_1 = \{1, 2, 3\}, F_1 = \{1, 3, 4\}, G_1 = \{2, 3, 4, 5\}$*; iteratively remove/add random 2-element subsets as described. Given* $E_1 = E_3$*, find* $P(S_1 = \{1, 2\})$.

Computing case-by-case for each of the three equally likely choices of S_1 (tracking which draws of S_2, S_3 are compatible with $E_3 = E_1$) gives $P(S_1 = \{1, 2\} \cap E_3 = E_1) = \frac{1}{60}$, and summing over all cases gives $P(E_3 = E_1) = \frac{1}{12}$. So $P(S_1 = \{1, 2\} \mid E_3 = E_1) = \frac{1/60}{1/12} = \frac{1}{5}$.
Answer: (A) 1/5

Q.4

$\theta_1 + \cdots + \theta_10 = 2\pi$ (positive angles); $z_k = e^{i(\theta_1+\cdots+\theta_k)}$. $P : \sum |z_{k+1} - z_k| \leq 2\pi$ (cyclic). $Q : \sum |z_{k+1}^2 - z_k^2| \leq 4\pi$ (cyclic).

$P = 2\sum \sin(\theta_k/2)$; by Jensen's inequality (concavity of sin on $(0,\pi)$), $\sum \sin(\theta_k/2) \leq 10\sin(\pi/10) \approx 3.09 < \pi$, so $P < 2\pi$: true. $Q = 2\sum |\sin\theta_k| \leq 2\sum \theta_k = 4\pi$ (using $|\sin x| \leq x$), with strict inequality: true.

Answer: (C) both P and Q are TRUE

Q.5

Three numbers chosen with replacement from $\{1, \ldots, 100\}$. $p_1 = P(\max \geq 81)$. Find $\frac{625}{4}p_1$.

$p_1 = 1 - (0.8)^3 = 1 - \frac{64}{125} = \frac{61}{125}$. $\frac{625}{4}p_1 = \frac{625}{4} \cdot \frac{61}{125} = \frac{305}{4} = 76.25$.

Answer: 76.25

Q.6

[Same setup] $p_2 = P(\min \leq 40)$. Find $\frac{125}{4}p_2$.

$p_2 = 1 - (0.6)^3 = 1 - \frac{27}{125} = \frac{98}{125}$. $\frac{125}{4}p_2 = \frac{98}{4} = 24.50$.

Answer: 24.50

Q.7

$x + 2y + 3z = \alpha, 4x + 5y + 6z = \beta, 7x + 8y + 9z = \gamma - 1$ consistent. $M = \begin{pmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. Find $|M|$.

Row-reducing the system for consistency gives the plane condition $\alpha - 2\beta + \gamma = 1$. Expanding $|M|$ along the third row gives $|M| = \gamma + \alpha - 2\beta = \alpha - 2\beta + \gamma = 1$.

Answer: 1

Q.8

[Same setup] P: the plane $\alpha - 2\beta + \gamma = 1$; D = squared distance from $(0, 1, 0)$ to P.

$$\text{Distance} = \frac{|0 - 2 - 1|}{\sqrt{1 + 4 + 1}} = \frac{3}{\sqrt{6}}, \text{ so } D = \frac{9}{6} = 1.5.$$

Answer: 1.50

Q.9

$L_1 : x\sqrt{2} + y - 1 = 0$, $L_2 : x\sqrt{2} - y + 1 = 0$; C = locus where (dist. to L_1)(dist. to L_2)$= \lambda^2$. Line $y = 2x + 1$ meets C at R, S with $RS = \sqrt{270}$. Find λ^2.

The product-of-distances condition gives $|2x^2 - (y-1)^2| = 3\lambda^2$. Substituting $y = 2x+1$ into the branch $(y - 1)^2 - 2x^2 = 3\lambda^2$ gives $2x^2 = 3\lambda^2$; the chord length works out to $2\sqrt{15\lambda^2/2} = \sqrt{270} \Rightarrow \lambda^2 = 9$.

Answer: 9.00

Q.10

[Same setup] Perpendicular bisector of RS meets C at R', S'; $D = (R'S')^2$.

The perpendicular bisector $y = 1 - x/2$ meets the other branch $2x^2 - (y-1)^2 = 27$ at $x^2 = \frac{108}{7}$; computing $(\Delta x)^2 + (\Delta y)^2$ for the two intersection points gives $D = \frac{540}{7} \approx 77.14$.
Answer: 77.14

Q.11

$E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{pmatrix}$. *$Q$ nonsingular* 3×3.

Swapping rows 2,3 and columns 2,3 of E reproduces F exactly, and $P^2 = I$: (A) true. Since $\det E = 0$ (direct expansion), $F = PEP$ is also singular; substituting $F = PEP$ shows $EQ + PFQ^{-1} = E(Q + PQ^{-1})$, and both sides of (B) evaluate to 0 using $\det E = 0$: true. Since $\det(EF) = \det E \cdot \det F = 0$, both sides of (C) are 0, so the strict inequality fails: false. Using $P^{-1} = P$ and $PEP = F$, $PFP = E$, both traces in (D) equal $2\,\mathrm{tr}(E) = 44$: true.
Answer: (A), (B), (D)

Q.12

$f(x) = \dfrac{x^2 - 3x - 6}{x^2 + 2x + 4}$. *Which statements are true?*

$f'(x) = \dfrac{5x(x+4)}{(x^2+2x+4)^2}$, negative on $(-4, 0)$, positive outside. So f is decreasing on $(-2, -1) \subset (-4, 0)$ (A true) and increasing on $(1, 2)$ (B true). Critical values: $f(-4) = \frac{11}{6}$ (local max), $f(0) = -\frac{3}{2}$ (local min); with horizontal asymptote $y = 1$, the range is exactly $\left[-\frac{3}{2}, \frac{11}{6}\right]$ – bounded, so f is not onto (C false) and the range is not $[-\frac{3}{2}, 2]$ (D false).
Answer: (A), (B)

Q.13

$P(E) = \frac{1}{8}, P(F) = \frac{1}{6}, P(G) = \frac{1}{4}, P(E \cap F \cap G) = \frac{1}{10}$.

(A) $P(E \cap F \cap G^c) = P(E \cap F) - \frac{1}{10} \le P(E) - \frac{1}{10} = \frac{1}{8} - \frac{1}{10} = \frac{1}{40}$: true. (B) $P(E^c \cap F \cap G) \le P(F \cap G) - \frac{1}{10} \le P(F) - \frac{1}{10} = \frac{1}{6} - \frac{1}{10} = \frac{1}{15}$: true. (C) By Boole's inequality, $P(E \cup F \cup G) \le P(E) + P(F) + P(G) = \frac{13}{24}$: true. (D) Since $P(E \cup F \cup G) \le \frac{13}{24} < \frac{7}{12}$, the complement $P(E^c \cap F^c \cap G^c) = 1 - P(E \cup F \cup G) > 1 - \frac{13}{24} = \frac{11}{24} > \frac{5}{12}$: false.
Answer: (A), (B), (C)

Q.14

E, F are 3×3; $I - EF$ invertible, $G = (I - EF)^{-1}$.

The push-through identity gives $(I-FE)^{-1} = I + FGE$, directly confirming (B) $(I-FE)(I+FGE) = I$. Testing the scalar analogue confirms (A) $|FE| = |I-FE||FGE|$ algebraically. From $G(I-EF) = I$ and $(I-EF)G = I$, both GEF and EFG equal $G-I$, confirming (C). Statement (D) uses the wrong sign and does not follow from the identity: false.

Answer: (A), (B), (C)

Q.15

$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\dfrac{1+k(k+1)x^2}{x}\right)$.

Using $\cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right) = \tan^{-1}((k+1)x) - \tan^{-1}(kx)$, the sum telescopes: $S_n(x) = \tan^{-1}((n+1)x) - \tan^{-1}(x)$. (A) For $n = 10$, this equals $\cot^{-1}\left(\frac{1+11x^2}{10x}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$: true (verified via tangent subtraction). (B) As $n \to \infty$, $S_n(x) \to \frac{\pi}{2} - \tan^{-1} x$, so $\cot(S_n(x)) \to x$: true. (C) $S_3(x) = \pi/4$ requires $3x/(1+4x^2) = 1$, i.e. $4x^2 - 3x + 1 = 0$, discriminant $= -7 < 0$: no real root, false. (D) $\tan(S_n(x)) = \frac{nx}{1+(n+1)x^2}$ has maximum $\frac{n}{2\sqrt{n+1}}$ over x; for $n = 2$ this is $\frac{1}{\sqrt{3}} \approx 0.577 > \frac{1}{2}$: false.

Answer: (A), (B)

Q.16

α, β real; for all z with $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, (x, y) lies on $x^2 + y^2 + 5x - 3y + 4 = 0$.

The points $z = -\alpha, -\beta$ must lie on the given circle at the endpoints of the chord subtending the given angle; since α, β are real, $-\alpha, -\beta$ are the circle's x-intercepts. Setting $y = 0$: $x^2 + 5x + 4 = 0 \Rightarrow x = -1, -4$, giving $\{\alpha, \beta\} = \{1, 4\}$.

Answer: (B) $\alpha\beta = 4$**, (D)** $\beta = 4$

Q.17

Number of real roots of $3x^2 - 4|x^2 - 1| + x - 1 = 0$.

For $|x| \geq 1$: $-x^2 + x + 3 = 0 \Rightarrow x = \frac{1 \pm \sqrt{13}}{2} \approx 2.30, -1.30$ (both valid). For $|x| < 1$: $7x^2 + x - 5 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{141}}{14} \approx 0.78, -0.92$ (both valid). Total: 4 real roots.

Answer: 4

Q.18

$\triangle ABC$: $AB = \sqrt{23}, BC = 3, CA = 4$. Find $\dfrac{\cot A + \cot C}{\cot B}$.

Using $\cot A = \frac{b^2+c^2-a^2}{4 \cdot \text{Area}}$ (and cyclic) with $a = 3, b = 4, c = \sqrt{23}$: $\cot A = \frac{30}{4K}$, $\cot B = \frac{16}{4K}$, $\cot C = \frac{2}{4K}$. Ratio $= \frac{30+2}{16} = 2$.

Answer: 2

Q.19

Unit vectors $\vec{u}, \vec{v}$ (not perpendicular), $\vec{w}$ with $\vec{u}\cdot\vec{w} = 1, \vec{v}\cdot\vec{w} = 1, \vec{w}\cdot\vec{w} = 4$. Volume of parallelepiped $= \sqrt{2}$. Find $|3\vec{u} + 5\vec{v}|$.

With $k = \vec{u}\cdot\vec{v}$, Volume$^2 = \det\begin{pmatrix} 1 & k & 1 \\ k & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix} = 2 + 2k - 4k^2 = 2 \Rightarrow k(1-2k) = 0$. Since $\vec{u}, \vec{v}$ not perpendicular, $k = \frac{1}{2}$. Then $|3\vec{u} + 5\vec{v}|^2 = 9 + 30(\frac{1}{2}) + 25 = 49 \Rightarrow |3\vec{u} + 5\vec{v}| = 7$.

Answer: 7