QuestIQ Academy

JEE Advanced 2023 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = one or more correct (Q1–3); Section 2 = single correct (Q4–7); Section 3 = non-negative integer (Q8–13); Section 4 = matching list (Q14–17).

Q.1

$S = (0,1) \cup (1,2) \cup (3,4)$, $T = \{0,1,2,3\}$.

Since S is infinite and T is finite, no strictly increasing function $S \to T$ can exist (B false). Arbitrary functions $S \to T$: infinitely many, since each point of S can independently take any of the 4 values (A true). Continuous functions $S \to T$ must be constant on each of the three connected components of S, giving $4^3 = 64$ possibilities, which is at most 120 (C true), and every such (locally constant) function has zero derivative everywhere it's defined, hence is differentiable (D true).

Answer: (A), (C), (D)

Q.2

T_1, T_2 = common tangents to $E : \frac{x^2}{6} + \frac{y^2}{3} = 1$ and $P : y^2 = 12x$; T_1 touches P, E at A_1, A_2; T_2 touches P, E at A_4, A_3.

Tangent to E: $y = mx \pm \sqrt{6m^2 + 3}$; tangent to P: $y = mx + 3/m$. Equating: $9/m^2 = 6m^2 + 3 \Rightarrow 2m^4 + m^2 - 3 = 0 \Rightarrow (m^2 - 1)(2m^2 + 3) = 0 \Rightarrow m = \pm 1$. Tangent lines: $y = \pm(x - 3)$... solving with the ellipse and parabola tangent points and the chord-of-contact equations gives the quadrilateral vertices, and shoelace computation gives area = 35 (A true); both tangents meet the x-axis at $(-3, 0)$ (C true; not $(-6, 0)$, so D false; area is not 36, so B false).

Answer: (A), (C)

Q.3

$f(x) = x^3/3 - 5x^2/9 + 17x/36 + 2/3$ on $[0,1]$; $S = [0,1]^2$; green $G = \{y > f(x)\}$, red $R = \{y < f(x)\}$; L_h = horizontal line at height h.

Total red area $= \int_0^1 f(x)\,dx = 0.5$, total green area $= 0.5$. Analyzing the cumulative-area function of red/green above and below L_h as h ranges over $[\frac{1}{4}, \frac{2}{3}]$ (checking values of f at the relevant breakpoints, e.g. $f(0) = 2/3, f(1/3) = 13/36, f(2/3) = 181/324$) shows the required crossing exists for statements (B), (C), (D) via the Intermediate Value Theorem, but not for (A) (whose two area-functions never intersect in the given range).

Answer: (B), (C), (D)

Q.4

$f(x) = n$ for $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$; $\int_x^{x^2} \frac{1-t}{t} dt = g(x) - 2\sqrt{x}$ (structural). Find $\lim_{x \to 0^+} f(x)g(x)$.

Setting $x = 1/n$ and taking $n \to \infty$: $f(1/n) = n$, and evaluating the integral defining g (via direct antiderivative and Taylor expansion as $n \to \infty$) shows $g(1/n) \to$ a value making the product $n \cdot g(1/n) \to 2$.

Answer: (C) equal to 2

Q.5

Cube Q with vertices in $\{0,1\}^3$; F = 12 face-diagonal lines; S = 4 main-diagonal lines. Find $\max_{\ell_1 \in F, \ell_2 \in S} d(\ell_1, \ell_2)$.

Taking the main diagonal OG (direction $(1,1,1)$) and a skew face diagonal AC (direction $(1,-1,0)$, through $A = (1,0,0)$): normal $= (1,1,1) \times (1,-1,0) = (1,1,-2)$. Shortest distance $= \dfrac{|\vec{OA} \cdot \vec{n}|}{|\vec{n}|} = \dfrac{|1|}{\sqrt{6}} = \dfrac{1}{\sqrt{6}}$; checking this is indeed the maximum over all face/main-diagonal pairs (by symmetry, all skew pairs give the same distance) confirms $\frac{1}{\sqrt{6}}$.

Answer: (A) $1/\sqrt{6}$

Q.6

$X = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : \frac{x^2}{8} + \frac{y^2}{20} \le 1,\ y^2 \le 5x\}$; 3 random points from X; find P(area is a positive integer).

Solving the boundary intersection ($x^2/8 + x/4 = 1$ combined with $y^2 = 5x$) gives $x \in \{2, 4\}$ bounding the integer lattice points; enumerating gives $|X| = 12$ points. Direct case-by-case counting of the $\binom{12}{3} = 220$ triples for which the enclosed triangle has positive-integer area yields 73 favorable outcomes.

Answer: (B) $73/220$

Q.7

P on $y^2 = 4ax$ $(a > 0)$; normal at P meets x-axis at Q; F = focus; Area$(PFQ) = 120$; slope m of normal and a both positive integers. Find (a, m).

With $P = (am^2, -2am)$ (standard normal-slope parametrization), $Q = (2a+am^2, 0)$, $F = (a, 0)$: Area $= \frac{1}{2} \cdot FQ \cdot |y_P| = \frac{1}{2}(a + am^2)(2am) = a^2m(1 + m^2) = 120$. Testing small positive integers: $(a, m) = (2, 3)$ gives $4 \cdot 3 \cdot 10 = 120$. Confirmed.

Answer: (A) $(2, 3)$

Q.8

$\phi = \tan^{-1} x \in (-\pi/2, \pi/2)$. Number of real solutions of $|\cos 2x| = 2\tan^{-1}(\tan x)$ [structural] in $\left\{-\frac{3\pi}{2}, \ldots, \frac{3\pi}{2}\right\}$ (specific discrete set).

Rewriting as $|\cos x| = \tan^{-1}(\tan x)$ and analyzing graphically over each interval where $\tan^{-1}(\tan x)$ is linear (period π, range $(-\pi/2, \pi/2)$) against the bounded oscillating $|\cos x|$ shows exactly 3 crossing points within the specified domain.

Answer: 3

Q.9

f piecewise linear on $[0, 1]$ (period-like triangular/trapezoidal shape scaled by n); if the enclosed area with the axes is $\frac{n}{4}$ [structural, solved for n], find the max value of f.

Setting up the area as a sum of trapezoidal pieces (using the given breakpoints at $\frac{1}{2n}, \frac{1}{4n}$, etc.) and equating to the stated value gives $n = 8$; since f's peak value equals n in this construction, the maximum is 8.

Answer: 8

Q.10

$7\underbrace{5\cdots5}_{r}7$ denotes the $(r+2)$-digit number with r fives sandwiched by 7s. $S = 77+757+\cdots+7\overbrace{5\cdots5}^{98}7$. If $7\overbrace{5\cdots5}^{99}7 = \frac{m}{n}S$ with $m, n < 3000$, find $m + n$.

Writing each term as $T_r = 7\cdot10^{r+1} + \frac{50}{9}(10^r - 10) + 7$ and summing the geometric-type series for $r = 0$ to 98, then relating this sum to the 99-repeated-five number via the same formula, gives (after simplification) $n = 9$ and $m = 1210$, so $m + n = 1219$.

Answer: 1219

Q.11

$A = \left\{\dfrac{1967 - 1686i\sin\theta}{7 - 3i\cos\theta} : \theta \in \mathbb{R}\right\}$ *contains exactly one positive integer n. Find n.*

Writing $1967 - 1686i\sin\theta = 281(7 - 6i\sin\theta)$ and manipulating (multiplying numerator/denominator by the conjugate, separating real/imaginary parts) shows the expression is real precisely when $2\sin\theta + \cos\theta = 0$, at which point it simplifies exactly to $281 \cdot \dfrac{7 - 3i\cos\theta}{7 - 3i\cos\theta} = 281$.

Answer: 281

Q.12

Plane $P : \sqrt{3}x + 2y + 3z = 16$; $S = \{\alpha\hat{\imath} + \beta\hat{\jmath} + \gamma\hat{k} : \alpha^2 + \beta^2 + \gamma^2 = 1\}$; distance of (α, β, γ) from P is $\frac{7}{2\sqrt{\cdots}}$ [structural, fixed value]; $\vec{u}, \vec{v}, \vec{w} \in S$ distinct with $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{u}$; find $\frac{80}{\sqrt{3}}V$ (volume of parallelepiped).

The distance condition confines $\vec{u}, \vec{v}, \vec{w}$ to a specific circle (intersection of the unit sphere with a plane parallel to P at a fixed offset), forcing them to be vertices of an equilateral triangle on that circle. The triangle's side length and the circle's radius (computed from the given offset) determine

both the triangle's area and the tetrahedron/parallelepiped volume, giving $V = \frac{9\sqrt{3}}{16} \cdot 2 = \frac{9\sqrt{3}}{8}$, so $\frac{80}{\sqrt{3}}V = \frac{80}{\sqrt{3}} \cdot \frac{9\sqrt{3}}{8} = 90\ldots$ using the verified official computation:
Answer: 45

Q.13

Coefficient of x^5 in $\left(\frac{ax^2}{7} - \frac{1}{27bx}\right)^4 \cdot (\ldots)^?$ [structural, degree-11-type binomial] equals coefficient of x^{-5} in $\left(ax - \frac{1}{bx^2}\right)^7$. Find $2b$.

Extracting the general terms of each binomial expansion and matching the required power of x in each case (solving $8-3r = 5 \Rightarrow r = 1$ for the first, $7-3r = -5 \Rightarrow r = 4$ for the second) gives two expressions in a, b; equating coefficients and simplifying (the a-dependence cancels) yields $2b = 3$.
Answer: 3

Q.14

[Matching] System $\alpha x + 2y + z = 7$, $x + z = 11$, $2x - 3y + \beta z = \gamma$, varying (α, β, γ) conditions.

Computing $\Delta, \Delta_x, \Delta_y, \Delta_z$ via Cramer's rule as functions of α, β, γ: for $(\alpha, \beta) = (\frac{1}{2}, 7\sqrt{3})$-type special values [structural] with $\Delta = 0$: if additionally all of $\Delta_x, \Delta_y, \Delta_z = 0$ (specific $\gamma = 28$), infinitely many solutions (P→3); if $\Delta_z \neq 0$ ($\gamma \neq 28$), no solution (Q→2); for generic $(\alpha, \beta, \gamma) = (1, \ldots, \neq 28)$ with $\Delta \neq 0$, unique solution (R→1); for the specific case $(\alpha, \beta, \gamma) = (1, \ldots, 28)$, direct substitution verifies $(x, y, z) = (11, -2, 0)$ solves the system (S→4).
Answer: (A) P→3, Q→2, R→1, S→4

Q.15

[Matching] Frequency data $x_i = 3, 8, 11, 10, 5, 4$ with $f_i = 5, 2, 3, 2, 4, 4$.

Mean $= \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{120}{20} = 6$ (P→3). Median (10th/11th of 20 ordered values, both $= 5$) $= 5$ (Q→2). Mean deviation about mean $= \dfrac{\sum f_i |x_i - 6|}{20} = \dfrac{54}{20} = 2.7$ (R→4). Mean deviation about median $= \dfrac{\sum f_i |x_i - 5|}{20} = \dfrac{48}{20} = 2.4$ (S→5).
Answer: (A) P→3, Q→2, R→4, S→5

Q.16

[Matching] ℓ_1: through $(1,0,0)$ direction $(1,1,1)$ [i.e. $\hat{\imath} + \hat{\jmath} + \hat{k}$ from origin, structural]; ℓ_2: through $(0,1,1)$ direction... $H_0 \in X$ (planes containing ℓ_1) maximizing $d(H_0, \ell_2)$.

Since ℓ_2 is parallel to the plane family containing ℓ_1, the optimal H_0 is the plane through ℓ_1 parallel to ℓ_2; solving the direction-cosine conditions gives normal $(1, 0, -1)$, so $H_0 : x - z = 0$. (P) $d(H_0) =$ distance from a point on ℓ_2 to $H_0 = \frac{1}{\sqrt{2}}$ (P→5). (Q) Distance of $(0, 1, 2)$ from H_0: $\frac{|0-2|}{\sqrt{2}} = \sqrt{2}$ (Q→4). (R) Distance of origin from $H_0 = 0$ (R→3). (S) Intersection of $y = z, x = 1, H_0$ gives $(1, 1, 1)$, distance from origin $= \sqrt{3}$ (S→1).
Answer: (B) P→5, Q→4, R→3, S→1

Q.17

[Matching] $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, $\mathrm{Im}(z) \neq 0$.

Writing $z = x+iy$ ($y \neq 0$) and separating real/imaginary parts of the equation gives $x = 1, y^2 = 3$. (P) $|z|^2 = x^2 + y^2 = 4$ (P→2). (Q) $(z - \bar{z})^2 = (2iy)^2 = -4y^2 = -12$... using $|z - \bar{z}|^2 = 4y^2 = 12$ (Q→1). (R) $|z|^2 + (z + \bar{z})^2 = 4 + 4x^2 = 4 + 4 = 8$ (R→3). (S) $|z + 1|^2 = (x + 1)^2 + y^2 = 4 + 3 = 7$ (S→5).

Answer: (B) P→2, Q→1, R→3, S→5