Total lecture = 3.

Remainder & Factor Theorem.

$\boxed{\text{Lecture-I}}$

e.g.: Find the remainder when x^2-5x+7 is divided by $x-2$.

Sol^n: Let $f(x) = x^2-5x+7$.

$$\text{Remainder} := f(2) = (2)^2 - 5(2) + 7$$
$$= 4 - 10 + 7$$
$$= 1$$

$$\begin{array}{r|l} & x-3 \\ \hline x-2 & x^2-5x+7 \\ & \underline{x^2-2x} \\ & \quad -3x+7 \\ & \quad \underline{-3x+6} \\ & \qquad 1 \end{array}$$

$1 \rightarrow$ Remainder.

e.g.: Find the remainder when x^2-5x+7 is divided by $x-1$.

Sol^n: Let $f(x) = x^2-5x+7 \longrightarrow \text{Remainder} = f(1) = (1)^2 - 5(1) + 7$
$$= 8-5 = \underline{\underline{3}}.$$

$\rightarrow$ Euclid's Division Lemma.

$$\boxed{\text{Dividend} = \text{Divisor} \times \text{quotient} + \text{Remainder}.}$$

$$(x^2-5x+7) = (x-2) \times q + r$$

Substitute $x = 2$.

$$(2)^2 - 5(2) + 7 = \underbrace{(2-2)}_{0} \times q + r$$
$$4 - 10 + 7 = r \Rightarrow r = 1$$

$f(x) = x^2$

$f(1) = (1)^2 = 1$

$f(2) = (2)^2 = 4$

$x \longrightarrow \boxed{f} \longrightarrow f(x)$

$2 \longrightarrow x^2 \longrightarrow 4$

eg: Find the remainder when x^2-5x+7 is divided $x-3$.

Sol^n: Let $f(x) = x^2-5x+7$ then remainder $= f(3) = 3^2-5(3)+7$

$= 9-15+7$

$= 16-15 = 1$

Remainder & Factor theorem.

Quotient

$$3\,\overline{)\,15\,}\quad 5$$

Divisor $\rightarrow$ 3, Dividend $\leftarrow$ 15, Quotient $\leftarrow$ 5

15

0 $\rightarrow$ Remainder

$$3\,\overline{)\,16\,}\quad 5$$

15

1

$$\boxed{\text{Dividend} = \text{Divisior} \times \text{Quotient} + \text{Remainder.}}$$

Euclid's Division Lemma.

Observation.

$$f(x) = x^2 - 5x + 6$$

$$f(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0.$$

$\Rightarrow$ $x-2$ is a factor of $x^2 - 5x + 6$.

$$x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = x(x-2) - 3(x-2)$$

$$x^2 - 5x + 6 = (x-2)(x-3)$$

$$f(3) = (3)^2 - 5(3) + 6 = 9 - 15 + 6 = 0$$

$\Rightarrow$ $x-3$ is a factor of $x^2 - 5x + 6$.

Factor theorem.

For any polynomial $f(x)$ if $f(\alpha)=0$ then $(x-\alpha)$ is a factor of $f(x)$.

Ex: * $f(x) = x^3 - x^2 + x - 1$

$f(1) = (1)^3 - 1^2 + 1 - 1 = 0$

$\Rightarrow$ $x-1$ is a factor of x^3-x^2+x-1

$f(x) = x^3 - x^2 + x - 1$

$f(-1) = (-1)^3 - (-1)^2 + (-1) - 1 = -1 - 1 - 1 - 1$

$= -4.$

$x-(-1) = x+1$ is not a factor of x^3-x^2+x-1

* If we divide x^3-x^2+x-1 by $x+1$ then the Remainder $f(-1)$

Remainder Theorem.

If a polynomial $f(x)$ is divided by $(x-\alpha)$ then the Remainder is $f(\alpha)$.

Note:

Remainder

① $f(x) / (x-\alpha) \longrightarrow f(\alpha)$

$x-\alpha=0$

$x=\alpha$

② $f(x) / (x+\alpha) \longrightarrow f(-\alpha)$

$x+\alpha=0$

$\Rightarrow x=-\alpha$

③ $f(x) / (ax-b) \longrightarrow f(b/a)$

$ax-b=0$

$\Rightarrow \quad x=b/a$

④ $f(x) / (ax+b) \longrightarrow f(-b/a)$

$ax+b=0$

$\Rightarrow ax=-b \Rightarrow x=\frac{-b}{a}$.

$\boxed{\text{Lecture-II}}$

Examples.

1) Using Remainder theorem, find the remainder when

ⓐ $(2x^2-x+1)$ is divided by $(x-2)$

Sol^n: Let $f(x)=2x^2-x+1$.

If we divide $f(x)$ by $(x-2)$ then Remainder $f(2)$

Remainder $= f(2) = 2(2)^2-2+1 = 8-2+1 = \underline{\underline{7}}$ $\underline{\underline{\text{Ans.}}}$

(b) $(2x^3+5x^2-9x+1)$ is divided by $(x+3)$

Sol^n: Let $f(x) = 2x^3+5x^2-9x+1$

If we divide $f(x)$ by $(x+3)$ then the

Remainder $= f(-3) = 2(-3)^3 + 5(-3)^2 - 9(-3) + 1$

$= -54 + 45 + 27 + 1$

$= \underline{\underline{19}}$ Ans.

2) Using Remainder theorem, find the remainder when

(a) $(3x^2+5x-9)$ divided by $(3x+2)$

Solⁿ: Let $f(x) = 3x^2+5x-9$

Remainder := $f(-2/3)$ (by Remainder Theorem)

$$f(-2/3) = 3\left(\frac{-2}{3}\right)^2 + 5\left(\frac{-2}{3}\right) - 9$$

$$= \frac{\cancel{3} \times 4}{\cancel{9}\,3} - \frac{10}{3} - 9$$

$$= \frac{4}{3} - \frac{10}{3} - 9 = \frac{-6}{3} - 9$$

$$= -2 - 9 = \underline{\underline{-11}}$$

(b) $8x^3-16x^2+14x-5$ divided by $2x-1$.

Solⁿ: Remainder = $f(1/2) = 8\left(\frac{1}{2}\right)^3 - 16\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 5$

$f(x)$

$$= 8\left(\frac{1}{8}\right) - 16\left(\frac{1}{4}\right) + 14\left(\frac{1}{2}\right) - 5$$

$$= 1 - 4 + 7 - 5$$

$$= 8 - 9 = \underline{\underline{-1}} \text{ Ans.}$$

3) When divided by $(x-3)$, the polynomial x^3-px^2+x+6 & $2x^3-x^2-(p+3)x-6$ leave the same remainder. Find the value of p. Also, find the remainder in each case. (2010)

Sol^n: Let $f(x) = x^3-px^2+x+6$ &

$$g(x) = 2x^3-x^2-(p+3)x-6.$$

When $f(x)$ & $g(x)$ are divided by $(x-3)$ then the remainder are $f(3)$ & $g(3)$ [by Remainder theorem]

Given: $f(3) = g(3)$

$$\Rightarrow (3)^3-p(3)^2+3+6 = 2(3)^3-(3)^2-(p+3)3-6$$

$$\Rightarrow 27-9p+9 = 54-9-3p-9-6$$

$$\Rightarrow 36-9p = 30-3p \Rightarrow 6 = 6p$$

$$\Rightarrow \boxed{p=1} \text{ Ans.}$$

$$\text{Remainder} = f(3) = 36-9p = 36-9 = 27$$
$$= g(3) = 30-3p = 30-3 = 27 \text{ Ans.}$$

4) Find a if the two polynomials ax^3+3x^2-9 and $2x^3-5x+a$ leave the same remainder when divided by $x+3$. (2015)

Sol^n: Let $f(x) = ax^3+3x^2-9$ & $g(x) = 2x^3-5x+a$.

$\because$ Remainder are equal

$\therefore$ $f(-3) = g(-3)$

$\Rightarrow a(-3)^3+3(-3)^2-9 = 2(-3)^3-5(-3)+a$

$\Rightarrow -27a+27-9 = -54+15+a$

$\Rightarrow -27a+18 = -39+a$

$\Rightarrow 18+39 = a+27a \Rightarrow 28a = 57$

$$\Rightarrow \boxed{a = \frac{57}{28}}$$

Remainder $= f(-3) = -27\left(\frac{57}{28}\right)+18 =$

$g(-3) = -39+\frac{57}{28} =$

Lecture-III

Recall:- Factor theorem.

$(x-\alpha)$ is a factor of $f(x)$ then $f(\alpha)=0$

$(x+\alpha)$ — " — then $f(-\alpha)=0$

$(ax-b)$ — " — " — then $f\left(\frac{b}{a}\right)=0$

$(ax+b)$ — " — " — then $f\left(-\frac{b}{a}\right)=0$

1) Show that $(x-1)$ is a factor of $x^3-7x^2+14x-8$.

Hence, Completely factorise the expression.

Solⁿ:- Let $f(x)= x^3-7x^2+14x-8$.

$$f(1)=(1)^3-7(1)^2+14(1)-8$$

$$f(1)=1-7+14-8=0$$

$\Rightarrow$ $(x-1)$ is a factor of $x^3-7x^2+14x-8$.

$$
\begin{array}{r|l}
 & x^2-6x+8 \\
\hline
x-1 & x^3-7x^2+14x-8. \\
 & \underline{\cancel{x^3} - x^2} \\
 & \quad -6x^2+14x-8 \\
 & \quad \underline{-6x^2+6x} \\
 & \qquad\quad 8x-8 \\
 & \qquad\quad \underline{8x-8} \\
 & \qquad\qquad 0
\end{array}
$$

$$\Rightarrow x^3-7x^2+14x-8 = (x-1)(x^2-6x+8)$$

$$x^3-7x^2+14x-8 = (x-1)(x-2)(x-4)$$

$$x^2-6x+8 = x^2-4x-2x+8$$

$$= x(x-4)-2(x-4)$$

$$= (x-4)(x-2)$$

2) Show that $(x-1)$ is a factor of $x^3-7x^2+14x-8$.

Hence, Completely factorise the expression.

<u>Sol</u>: Let $f(x) = x^3-7x^2+14x-8$

$$f(1) = (1)^3 - 7(1)^2 + 14(1) - 8 = 1-7+14-8 = 0$$

$\Rightarrow$ $(x-1)$ is a factor of $x^3-7x^2+14x-8$

$$
\begin{array}{r|l}
 & x^2-6x+8 \\
\hline
x-1 & x^3-7x^2+14x-8 \\
 & -x^3 \mp x^2 \\
\hline
 & -6x^2+14x-8 \\
 & -6x^2+6x \\
\hline
 & 8x-8 \\
 & 8x \mp 8 \\
\hline
 & 0
\end{array}
$$

$$
\begin{array}{r|l}
 & 55 \\
\hline
3 & 165 \\
 & 15 \\
\hline
 & 15 \\
 & -15 \\
\hline
 & 0
\end{array}
$$

$165 = 3 \times 55$.

$\Rightarrow$ $x^3-7x^2+14x-8 = (x-1)(x^2-6x+8)$

$$x^2-6x+8 = x^2-4x-2x+8$$
$$= x(x-4)-2(x-4) = (x-4)(x-2)$$

$$\boxed{x^3-7x^2+14x-8 = (x-1)(x-2)(x-4)}$$

14) Using the Remainder Theorem, find the remainder when $x^3 + (kx+8)x + k$ is divided by $x+1$ & $x-2$.

Hence find k if the sum of the two remainders is 1.

Sol: Let $f(x) = x^3 + (kx+8)x + k = x^3 + kx^2 + 8x + k$

Remainder: $f(x)/(x+1) \rightarrow f(-1) = (-1)^3 + k(-1)^2 + 8(-1) + k$

$= -1 + k - 8 + k$

$f(-1) = 2k - 9$

Remainder: $f(x)/(x-2) \rightarrow f(2) = (2)^3 + k(2)^2 + 8(2) + k$

$= 8 + 4k + 16 + k$

$f(2) = 5k + 24$

Given: $f(-1) + f(2) = 1 \Rightarrow 2k - 9 + 5k + 24 = 1$

$\Rightarrow 7k + 15 = 1 \Rightarrow 7k = -14$

$\Rightarrow \boxed{k = -2}$ Ans.

2) Using factor theorem, Show that $(3x-2)$ is a factor of $3x^3+x^2-20x+12$ & Hence factorise it Completely.

Solⁿ: Let $f(x) = 3x^3 + x^2 - 20x + 12$.

$$f\left(\tfrac{2}{3}\right) = 3\left(\tfrac{2}{3}\right)^3 + \left(\tfrac{2}{3}\right)^2 - 20\left(\tfrac{2}{3}\right) + 12$$

$$= \cancel{3}\left(\frac{8}{\cancel{27}_{\,9}}\right) + \frac{4}{9} - \frac{40}{3} + 12$$

$$= \frac{8 + 4 - 120 + 108}{9}$$

$$= \frac{120 - 120}{9} = 0$$

$\Rightarrow$ $3x-2$ is a factor of $3x^3 + x^2 - 20x + 12$.

$$
\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
3x-2 & 3x^3 + x^2 - 20x + 12 \\
 & \underline{-\;3x^3 \mp 2x^2} \\
 & \quad 3x^2 - 20x + 12 \\
 & \quad \underline{-\;3x^2 \mp 2x} \\
 & \qquad -18x + 12 \\
 & \qquad \underline{-\;(-18x \pm 12)} \\
 & \qquad\qquad 0
\end{array}
$$

$\Rightarrow$ $3x^3 + x^2 - 20x + 12 = (3x-2)(x^2+x-6) = (3x-2)(x+3)(x-2)$

3) If x^3-2x^2+ax+b has a factor $(x+2)$ & leaves a remainder 9 when divided by $(x+1)$, find the values of a & b. With these values of a & b. factorise the given polynomial completely.

Solⁿ: Let $f(x) = x^3-2x^2+ax+b$.

* $\because$ $x+2$ is a factor of $f(x)$ $\therefore$ $f(-2)=0$

$(-2)^3-2(-2)^2+a(-2)+b=0$

$-8-8-2a+b=0 \Rightarrow \boxed{-2a+b=16}$

** $\because$ Remainder of $f(x)$ when divided by $x+1$ is 9

$\therefore$ $f(-1)=9 \Rightarrow (-1)^3-2(-1)^2+a(-1)+b=9$

$\Rightarrow -1-2-a+b=9$

$\boxed{-a+b=12}$

$$\begin{aligned} -\quad -2a+b&=16\\ +a\pm b&=-12\\ \hline -a&=4 \end{aligned} \Rightarrow \boxed{a=-4}$$

$-(-4)+b=12$

$\boxed{b=8}$

$$
\begin{array}{r|l}
 & x^2-4x+4 \\
\hline
x+2 & x^3-2x^2-4x+8 \\
 & \underline{-x^3 \mp 2x^2} \\
 & -4x^2-4x+8 \\
 & \underline{\mp 4x^2 \mp 8x} \\
 & 4x+8 \\
 & \underline{-4x \mp 8} \\
 & 0
\end{array}
$$

$$\Rightarrow (x)^2 - 2(2)(x) + (2)^2.$$

$$\Rightarrow \quad x^3-2x^2-4x+8 = (x+2)(x^2-4x+4)$$
$$= (x+2)(x-2)^2 \quad \underline{\underline{\text{Ans.}}}$$

Hit and Trial Method.

1) Use the factor theorem to factorise Completely

$$f(x) = x^3+x^2-4x-4.$$

Sol: Let $f(x) = x^3+x^2-4x-4.$

Substitute $x=1$, $f(1) = (1)^3+(1)^2-4-4 = -6$

Substitute $x=-1$, $f(-1) = (-1)^3+(-1)^2-4(-1)-4$

$$= -1+1+4-4 = 0$$

$\Rightarrow$ $x+1$ is a factor of $f(x)$.

$$
\begin{array}{r|l}
 & x^2 - 4 \\
\hline
(x+1) & x^3 + x^2 - 4x - 4 \\
 & \underline{-x^3 \mp x^2} \\
 & \quad\quad -4x - 4 \\
 & \quad\quad \underline{\mp 4x \mp 4} \\
 & \quad\quad\quad 0
\end{array}
$$

$$x^3 + x^2 - 4x - 4 = (x+1)(x^2-4) = (x+1)(x-2)(x+2)$$

Ans.

2) Using factor theorem, factorise completely.

$$f(x) = 2x^3 + 19x^2 + 38x + 21$$

Sol:
$$f(-1) = 2(-1)^3 + 19(-1)^2 + 38(-1) + 21$$
$$= -2 + 19 - 38 + 21 = 0$$

$\Rightarrow$ $x+1$ is a factor of $f(x)$

$$
\begin{array}{r|l}
 & 2x^2 + 17x + 21 \\
\hline
x+1 & 2x^3 + 19x^2 + 38x + 21
\end{array}
$$

$$f(x) = (x+1)(2x^2 + 17x + 21)$$
$$= (x+1)(2x^2 + 14x + 3x + 21)$$
$$= (x+1)(2x(x+7) + 3(x+7))$$
$$= (x+1)(x+7)(2x+3) \quad \text{Ans.}$$